Math 108a	Professor: Padraic Bartlett

Homework 9: Volume and Determinants

Due Friday, 12/6/13, 1:30 pm — *UCSB 2013*

As always, problems need to show work and proofs in order to receive full credit. Have fun! Also, there are a pair of bonus questions at the end of this problem set! Check them out.

1. Find the volume of the following paralleletopes:

   (a) $\{(1,2),(4,1)\}$ (b) $\{(1,1,1),(1,1,0),(1,0,0)\}$.

2. Find the positive determinants of the following matrices:

   (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

3. Find the determinants (i.e. not the positive determinant, the normal determinant, that we defined on Wednesday / in the online notes) of the following matrices:

   (a) $\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

4. Suppose that A is a matrix of the form

$$\begin{bmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

   Show, only using the techniques that we've studied in class thus far, that the determinant of A is 1.

5. For any n, define the $n \times n$ **checkerboard matrix** C_n as the following matrix: $c_{ij} = 1$ if $i + j$ is even, and 0 otherwise. For example, the 5×5 checkerboard matrix is the following object:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

   For any n, find the value of $\det(C_n)$.

1 Bonus Problems!

1. Determinant Tic-Tac-Toe is the following game:

   - There are two players, player 0 and player 1. They play using a 3×3 matrix, which starts off with all of its cells blank.
   - Player 1 places a 1 into some cell in an empty 3×3 matrix.
   - Player 0 counters with a 0 in a vacant position.
   - Play continues back and forth, with player 1 entering 1's and player 0 entering 0's, until the 3×3 matrix is completely filled in with five 1's and four 0's.
   - Player 0 wins if the determinant is 0; player 1 wins otherwise.

   (a) (+5 pts) Beat me in a game of determinant Tic-Tac-Toe. (You may choose to go first or second. Games can either be played in person during OH, before/after class, or over email. One attempt per person; if you lose, no do-overs. Games can be played until Tuesday, December 10.)

   (b) (+5 pts) Can player 1 always win; i.e. no matter what player 0 does, can player 1 always force the determinant to be 0? Either create a winning strategy for player 1 if it exists, or create a "response" strategy for player two that can force player 1 to lose, no matter what they do. Prove that your strategy works.

2. Let A be a $n \times n$ matrix with entries $a_{ij} = |i - j|$. For example, when $n = 3$, we have

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

   (+10 pts) Find the determinant of A for arbitrary values of n.