Math 108a | Professor: Padraic Bartlett

Homework 6: Linear Maps: Odds and Ends

Due Friday, 11/8/13, 1:30 pm | *UCSB 2013*

Remember: homework problems need to show work in order to receive full credit. Simply stating an answer is only half of the problem in mathematics; you also need to include an argument that persuades your audience that your answer is correct! As always, if you have any questions, feel free to contact either Shahab or I via email or office hours. Have fun!

1. Let $T_1 : U_1 \to U_2, T_2 : U_2 \to U_3, \ldots T_n : U_n \to U_{n+1}$ be some collection of injective linear maps. Look at the composition $T_n \circ T_{n-1} \circ \ldots \circ T_2 \circ T_1$, which is a map from U_1 to U_{n+1}. Prove that $T_n \circ T_{n-1} \circ \ldots T_2 \circ T_1$ is injective.

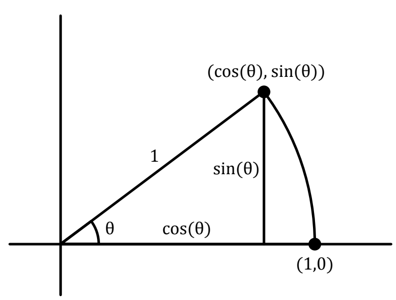
2. Let $T_1 : U_1 \to U_2, T_2 : U_2 \to U_3, \ldots T_n : U_n \to U_{n+1}$ be some collection of surjective linear maps. Look at the composition $T_n \circ T_{n-1} \circ \ldots \circ T_2 \circ T_1$, which is a map from U_1 to U_{n+1}. Prove that $T_n \circ T_{n-1} \circ \ldots T_2 \circ T_1$ is surjective.

3. Let $S = \{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ be some set of vectors, and let $\text{span}(S) = V$ denote the vector space spanned by these vectors. Let $T : V \to V$ be a surjective linear map. Show that $T(S) = \{T(\vec{v_1}), T(\vec{v_2}), \ldots T(\vec{v_n})\}$ also spans V.

4. Let $T : \mathbb{R}^5 \to \mathbb{R}^2$ be a linear map with nullspace $\text{null}(T) = \{(a, b, c, d, e) \mid a = b, a + b + c + d + e = 0 \in \mathbb{R}\}$. Prove that T is surjective.

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^7$ be a linear map with $\text{range}(T) = \{(a, b, c, d, e, f, g) \mid a + b + c = 0, d + e + f = 0, g = 0\}$. Prove that T is injective.

6. Consider the map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, that takes a vector (x, y) and rotates it by angle θ in a counterclockwise direction around the origin. For example, the vector $(1, 0)$ gets mapped to $(\cos(\theta), \sin(\theta))$, as depicted below:



   (a) Show that the map T_θ is linear.
   (b) Find coefficients $\alpha, \beta, \gamma, \delta$ such that
   $$T(x, y) = (\alpha x + \beta y, \gamma x + \delta y).$$

7. How much time did this set take? (As always, asked for calibration purposes.)