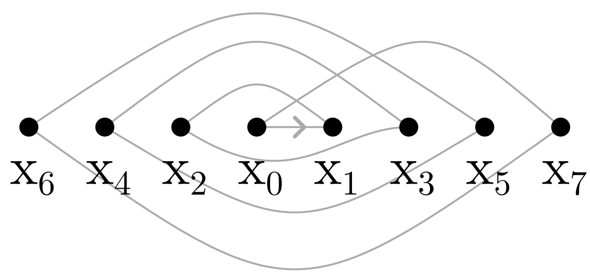
Lecture 3: The Alternating Points Lemma

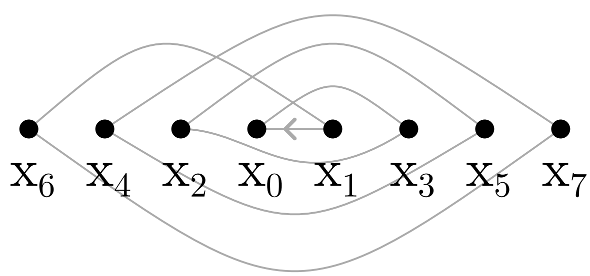
We ended last class with our statement of the following theorem:

Lemma. (Alternating Points Lemma.) Suppose that f is a continuous function on some interval I with the following two properties:

- There is some n such that f has a point, x_0, of period $2n+1$.
- For all $1 \leqslant m < n$, f has no points of period $2m+1$.

Then the orbit of x_0 must look like one of the following:





In other words: if we define $x_i = f^i(x_0)$, we have either

$$x_{2n} < x_{2n-2} < \ldots < x_4 < x_2 < x_0 < x_1 < x_3 < \ldots < x_{2n-1}, \text{ or}$$
$$x_{2n-1} < x_{2n-3} < \ldots < x_3 < x_1 < x_0 < x_2 < x_4 < \ldots < x_{2n}.$$

In the following section, we recap the motivation and notation we introduced in our last talk as well; from there, we move to the actual proof of our claim!

Proof. To make our notation easier, reorder the $2n+1$ points in the orbit of x_0 as the sequence

$$z_1 < z_2 < \ldots < z_{2n} < z_{2n+1}.$$

Let's look at what we're trying to prove. In both of the pictures that we're trying to prove must hold, we have this sort of "spiraling-out" relation, where

- an initially small interval $[x_0, x_1]$ becomes after one application of f the slightly larger interval $[x_2, x_1]$,
- which after another application of f becomes the larger interval $[x_2, x_4]$,
- which after yet another application of f becomes the larger interval $[x_3, x_4]$,
- ...
- which at our last stage becomes our entire collection of points!

Furthermore, this "spiraling-out" relation pretty much enforces the structure we're claiming must exist! Namely, suppose there is some seed interval $[z_m, z_{m+1}]$ such that the following holds:

- Repeatedly applying f to this interval expands it by precisely one z_i at each application.
- The "side" this new z_i shows up on alternates from the farthest-left to the farthest-right.

Then, we have the structure we want! (Sketch out anything satisfying the two properties above to see why this is sufficient.) This suggests that if we want to prove our claim, we should attempt to look for this sort of interval structure in our z_i's.

We start by introducing some notation. For any k, l, let S_{kl} denote the set

$$S_{kl} = \{z_i \ : \ k \leqslant i \leqslant l\}.$$

Moreover: given any such S_{kl}, if $\min_{z \in S_{kl}} f(z) = z_i, \max_{z \in S_{kl}} f(z) = z_j,$, we define

$$f(S_{kl}) = S_{ij}.$$

This is a convenient definition that will save us a lot of work in the coming pages.

With this notation established, let's start our proof. First, we want to find an appropriate "seed" interval $\{z_m, z_{m+1}\}$. Notice that in our pictures, this seed interval starts at the largest value of m such that $f(z_m) > z_m$. We know some such m exists, because $f(z_1) > z_1$; moreover, we know that $m \leqslant 2n$, because $f(z_{2n+1}) < z_{2n+1}$.

Now, let's look at what happens when this "seed" interval unfolds. Formally: define the sets S_{k_i, l_i}, which for shorthand we'll just refer to as S_i's, as follows: initialize $S_1 = \{z_m, z_{m+1}\}$, and if S_i exists and is not $S_{1,2n+1}$, define $S_{i+1} = f(S_i)$. This gives us a sequence of sets corresponding to repeated applications of f: we now want to show that they have the structure we described above!

First off, notice that for any i, $S_{i+1} \supsetneq S_i$. That these are not equal to each other is because the points z_i all correspond to an orbit of length $2n + 1$, and in particular if we apply f to any proper subset of these z_i's we should not be able to remain in that proper subset (as otherwise our orbit would be smaller than $2n + 1$!) That they are contained within each other is a straightforward induction proof:

Base case: To see that $S_2 = f(S_1) = f(\{z_m, z_{m+1}\})$ contains $\{z_m, z_{m+1}\}$, observe that that by construction, $f(z_m) \geqslant z_{m+1}$, while $f(z_{m+1}) \leqslant z_m$. Therefore the interval corresponding to S_2 must contain both z_m and z_{m+1}, as claimed.

Inductive step: Suppose that $S_i = f(S_{i-1}) \supset S_{i-1}$. Then simply notice that

$$S_{i+1} = f(S_i) \supset f(S_{i-1}) = S_i,$$

and thus that our inductive claim is proven.

We have constructed a sequence of intervals $S_i = f^{i-1}(S_1)$ that "expand" from some base interval S_1 until they cover everything, and grow by at least one element on each application of f. Let S_t be the last interval we get in this process: note that $S_t = S_{1,2n+1}$, our whole orbit.

We now want to show that this sequence grows by **exactly** one element on each application of f: or in other words, that our sequence $S_1, \ldots S_t$ has precisely $2n$ terms (i.e. $t = 2n$.) We prove this claim by contradiction: suppose not, that $t < 2n$. We will use this observation to create a point with odd period less than $2n + 1$, which will contradict our earlier claim!

To do this, form the following sequence of intervals: for each S_i, form the corresponding closed interval I_i that you get by taking all of the elements bounded between elements of S_i. Formally:

$$I_i = \{x \ : \ \text{there are } z_a, z_b \in S_i \text{ such that } z_a \leqslant x \leqslant z_b\}.$$

Define the following sequence of intervals: let

- $J_0, J_1, \ldots J_{2n-t-1} = I_1$,
- $J_{2n-t} = I_2$,
- $J_{2n-t+1} = I_3$,
- $\ldots$
- $J_{2n-2} = I_t$.

Notice that this sequence exists, is well-defined, and contains all of our intervals iff $t < 2n$. Also notice that we've proven above that

$$J_0 \to J_1 \to J_2 \to \ldots J_{2n-2},$$

and because $I_t = [z_1, z_{2n+1}]$, we trivially have $J_{2n-2} \to J_0$.

We want to use the itinerary lemma to create a periodic point with period $2n - 1$ via this sequence of $2n-1$ intervals! What we have here does... almost that. We definitely have the conditions needed to apply the itinerary lemma — we have f continuous and $J_i \to J_{i+1}$ for all $i \in \{0, \ldots 2n - 2\}$ (where we assume that we wrap around at $i = 2n - 2$), which is all we need to have to apply our lemma.

However, to get that our lemma will give us a point with period $2n-1$, we also need that our point doesn't come up anywhere earlier in our sequence! This is not something that our current set of intervals does for us. In particular, our last interval $J_{2n-2} = I_t = [z_1, z_{2n+1}]$ is our entire space — this is going to stop us from being able to form the kinds of contradictions we could make in our "period three implies chaos" proof," like where we deduced that a point being in $[x_0, x_1]$ and $[x_1, x_2]$ created problems for us.

So: we want a **better** I_t. Namely, a **smaller** I_t, that will still have the properties we want ($I_{t-1} \to I_t \to I_0$) but also create issues when we try to have points in both I_0 and I_t.

The smallest set we could hope for is one, say, of the form $[z_l, z_{l+1}]$, for some value of l. Can we find such a set?

Conveniently, we can! To see why, consider the two sets $\{z_1, \ldots, z_m\}$ and $\{z_{m+1}, \ldots, z_{2n+1}\}$. These two sets have different sizes; without loss of generality, assume for the moment that $m > 2n+1-m$ (the other case is handled in an identical fashion.) We know that, on one hand, there are some elements $z_i \in \{z_1, \ldots z_m\}$ such that $f(z_i) \notin \{z_1, \ldots z_m\}$; for example, $f(z_m) > z_m$, by definition!

On the other hand, because $m > 2n+1-m$, not **every** element of $\{z_1, \ldots z_m\}$ gets sent to $\{z_{m+1}, \ldots z_{2n+1}\}$. Consequently, there must be some value l such that $f(z_l) \in \{z_1, \ldots z_m\}$, and $f(z_{l+1}) \in \{z_{m+1}, \ldots z_{2n+1}\}$. In other words, $f(S_{l,l+1})$ contains $[z_m, z_{m+1}] = I_0$.

Using this discovery, let's redefine our stopping value t and our sequence $S_1, S_2, S_3\ ldots$ as follows: let $t-1$ denote the smallest value of i such that $S_i \to S_{l,l+1}$ holds. Such a value of t must exist, because the S_i's grow by at least one z_j at each step, and thus eventually span all of the periodic points in our interval (and in particular will eventually contain $S_{l,l+1}$.) Change S_t so that $S_t = S_{l,l+1} = \{z_l, z_{l+1}\}$. Finally, update the J_i's accordingly given our new values of t, S_t.

Notice that we still have

$$J_0 \to J_1 \to J_2 \to \ldots \to J_{2n-2} \to J_0,$$

as before. Therefore, we still have a sequence that we can apply the itinerary lemma to! Do so, and get a value y such that

- $f^k(y) \in J_k$, for all $0 \leqslant k \leqslant 2n-2$.
- $f^{2n-1}(y) = y$.

We know that y cannot be a point of period $2n-1$, because (by definition) our function has no points of odd period smaller than $2n-1$. Consequently, there must be some value of k such that $f^k(y) = y$, for $k < 2n-1$.

But this means, in particular, that

$$f^{2n-2}(y) = f^{2n-2-k}(y),$$

and thus that whatever value this is lies in the intersection of $J_{2n-2} = I_t$ and $J_{2n-2-k} = I_{t-k}$. But is this possible?

Well: recall that we defined S_{t-1} to be the first set such that $f(S_{t-1}) \supset S_{l,l+1}$. In particular, because all of the S_i's are nested (as proven earlier,) this means that for any $1 \leqslant i \leqslant t-1$, S_i does not properly contain $S_{l,l+1}$!

Therefore, the intersection of $S_{l,l+1}$ and any S_i for $1 \leqslant i \leqslant t-1$ must contain at most one point; in particular, one of the endpoints of $S_{l,l+1}$. Consequently, because the intervals J_{2n-2}, J_{2n-2-k} consist simply of the values between these endpoints, we know that the intersection of J_{2n-2} and J_{2n-2-k} must consist of at most one point, which is one of our z_i's.

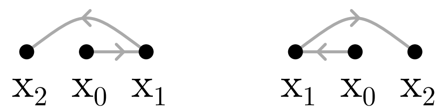
In particular, this means that there is some i such that

$$f^{2n-2}(y) = f^{2n-2-k}(y) = z_i.$$

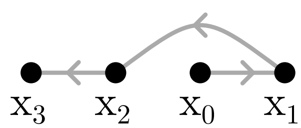
But this implies that $f^{2n-1}(z_i) = z_i$, because it is a point on an orbit of some length dividing $2n-1$. This contradicts the fact that the z_i's correspond to an orbit of length $2n+1$. Therefore our original assumption, that $t < 2n$, must have been flawed!

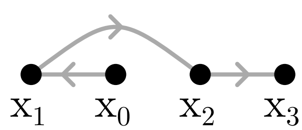
So we have that there are $2n$ sets $S_1, \ldots S_{2n}$, each contained within the other, and each precisely one larger than the other. All we need now, then, is the "alternating" condition that we claimed from before: i.e. each time we expand from S_i to S_{i+1}, the new point we add is always an endpoint of the corresponding I_{i+1}, occurs on the opposite side from the second-most-recently added point,

This is not hard to see. First, notice that it holds when we go from S_1 to S_2: because $f(z_m) > z_m, f(z_{m+1}) < z_{m+1}$, we are always in one of the following two cases:

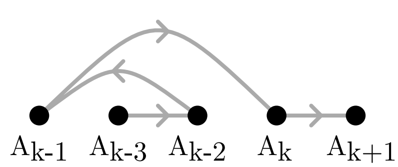


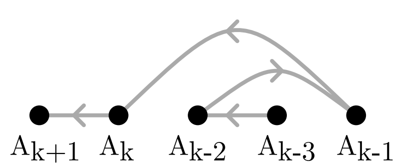
Showing that it holds when we go from S_2 to S_3 is a similar edge case / part of the homework. Essentially, you should show that any situation like the following is impossible for our function f:





Assuming that you have done this, we are left with just the task of analyzing the transition from S_i to S_{i+1}, for $i \geqslant 3$. Suppose for contradiction that our proof fails here, but has held up until this point. In this situation, if A_i denotes the unique element in $S_i \backslash S_{i-1}$, we must be in one of the following two pictures:





Notice that in either of these pictures, we have

$$\circlearrowleft [A_{k-2}, A_k] \leftrightarrows [A_{k-1}, A_{k-3}],$$

and thus can apply the itinerary lemma to deduce the existence of a point of period 3, which gives us a contradiction to our claim that we did not have points of odd order less than $2n+1$.

This then concludes our proof! We have created a sequence of nested intervals $S_1, \ldots S_{2n}$ that grow by one at each step, such that the new elements A_i added to each interval always occur on the furthest-left and furthest-right of these S_i's, in alternating order. $\square$