Dynamical Systems | Instructor: Padraic Bartlett

Lecture 2: Period Three Implies Chaos

Week 3 | *Mathcamp 2014*

We ended our last class by proving the following result (whose name I found while doing some research last night):

Theorem. (The Itinerary Lemma.) Let $f(x)$ be a continuous function on the interval $[a, b]$, and $I_0, \ldots I_{n-1}$ denote a collection of closed intervals that are each contained within $[a, b]$. Assume that

1. $f(I_k) \supseteq I_{k+1}$, for every $k = 0 \ldots n-2$, and
2. $f(I_{n-1}) \supseteq I_0$,

where by $f(I_k)$ we mean the set given by applying f to all of the points in the interval I_k. (In other words, applying f to any one interval I_k gives you a set that contains the next interval I_{k+1})

Then there is some point $x_0 \in I_0$ such that

1. $f^n(x_0) = x_0$, and
2. $f^k(x_0) \in I_k$, for every $k = 0, \ldots n-1$.

Today, we're going to talk about how to use this result!

1 Why We Care: Chaos

Consider the following problem:

Problem. Suppose we have a fluid filled with particles in some reasonably-close-to-one-dimensional object, which we can model as an interval $[a, b]$. Furthermore, suppose that we know how this fluid is "mixing:" i.e. that we have some function $f : [a, b] \to [a, b]$, such that $f(x)$ tells you where a particle at location x will wind up after one step forward in time.

Where do your particles go? Do they settle down? Do they all clump together at one end? In other words: what does f^n look like as n grows very large?

Something you might hope for is that your fluid particles settle down: that they either converge to various states, or at least that they all settle into some small set of predictable periodic orbits. In the worst case scenario, however, you might have something like the following:

Definition. A function f is called **chaotic** if for any n, it has a particle of period n.

So! The punchline for this class, and the reason we proved the Itinerary Lemma, is the following theorem of Li and Yorke:

Theorem. Suppose that f is a continuous function on $[a, b]$ with range contained in $[a, b]$. Then if f has a 3-periodic point, it is chaotic.

Before we start this proof, we introduce some useful notation:

Notation. Given a function f and intervals I_0, I_1, we write $I_0 \to I_1$ if $f(I_0) \supseteq I_1$. In particular, if $f(I_0) \supset I_i$, we will write $I_0 \circlearrowleft$ to denote that f of this interval contains itself.

Proof. We first note that because our function is a continuous map on $[a, b]$ that contains $[a, b]$ in its range, we get a point of period 1 by the intermediate value theorem, as mentioned on day 1. The case for $k = 2$ is similarly trivial; we leave this for the HW! So it suffices to search for points of period k, for $k \geqslant 2$.

Take a triple $x_0 < x_1 < x_2$ of points that form a 3-periodic orbit. Either $f(x_1) = x_2$ or $f(x_1) = x_0$; assume that $f(x_1) = x_0$ without loss of generality, as the proof will proceed identically in the other case. Then we have $f(f(x_1)) = f(x_0) = x_2$.

Let $I_0^\star = [x_0, x_1]$ and $I_1^\star = [x_1, x_2]$. Note that because $f(x_0) = x_2, f(x_1) = x_0, f(x_2) = x_1$, by the intermediate value theorem, we have

$$I_1^* \leftrightarrows I_0^* \circlearrowleft .$$

So: let $I_0 = \ldots I_{n-2} = I_0^\star$, and $I_{n-1} = I_1^\star$. Apply our theorem from before that was designed to find periodic points: this gives us a point y such that $y, f(y), \ldots f^{n-2}(y) \in I_0^\star$, $f^{n-1}(y) \in I_1^\star$, and $f^n(y) = y$.

I claim that this point is a n-periodic point. We already have that $f^n(y) = y$; we just need to prove that $f^k(y) \neq y$, for any $k = 1, \ldots n-1$. To see this, proceed by contradiction. Suppose that $f^k(y) = y$, for some $k < n-1$. Then $f^{n-1}(y)$ is equal to an earlier term $f^{n-1-k}(y)$, because applying f k times is the same thing as doing nothing. But this means that

- on one hand, $f^{n-1}(y) \in I_1^\star$, and
- on the other hand, $f^{n-1}(y) = f^{n-1-k}(y) \in I_{n-1-k} = I_0^\star$.

Therefore this point is in both sets. But the only point in both $I_0^\star = [x_0, x_1]$ and $I_1^\star = [x_1, x_2]$ is x_1; so $f^{n-1}(y) = x_1$. But then $f^n(y) = y = x_0$, and thus $f(y) = f(x_0) = x_2 \notin I_1 = [x_0.x_1]$.

So we have a contradiction to our assumption that x_0 was not a point with period n. $\square$

This is ... weird. All we used in the above statement was that there was a point with period 3 – i.e. some point such that $f(f(f(x))) = x$, while $f(x), f(f(x) \neq x$. And out of nowhere we got points of **every** period: chaos!

Surprisingly, this result is not even the strangest thing we're proving in this talk. Consider the following ordering on the natural numbers:

Definition. The **Sharkovsky** ordering on the natural numbers is the following ordering:

$$\begin{aligned} & 2^0 \cdot 3 \lhd 2^0 \cdot 5 \lhd 2^0 \cdot 7 \lhd 2^0 \cdot 9 \lhd \ldots \lhd 2^1 \cdot 3 \lhd 2^1 \cdot 5 \lhd 2^1 \cdot 7 \lhd 2^1 \cdot 9 \lhd \ldots \\ \ldots \lhd\ & 2^2 \cdot 3 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 7 \lhd 2^2 \cdot 9 \lhd \ldots \lhd 2^3 \cdot 3 \lhd 2^3 \cdot 5 \lhd 2^3 \cdot 7 \lhd 2^3 \cdot 9 \lhd \ldots \\ \ldots & \\ \ldots \lhd\ & 2^5 \lhd 2^4 \lhd 2^3 \lhd 2^2 \lhd 2^1 \lhd 1. \end{aligned}$$

In words: take the natural numbers, and break them into the following groups:

- S_0: All odd numbers greater than 1.
- S_1: All numbers of the form $2^1 \cdot$ (an odd number greater than 1).
- S_2: All numbers of the form $2^2 \cdot$ (an odd number greater than 1).
- ...
- S_∞: All powers of two.

For each $i \in \mathbb{N}$, order via the Sharkovsky ordering $\lhd$ each S_i using the normal ordering on the natural numbers: i.e $3 \lhd 5 \lhd 7 \ldots$

Order the set S_∞ using the opposite ordering from the normal ordering on the natural numbers: i.e. $\ldots \lhd 16 \lhd 8 \lhd 4 \lhd 2 \lhd 1$.

Finally, to compare any two elements s_i, s_j from different sets S_i, S_j, simply say that $s_i \lhd s_j$ whenever $i < j$ in the normal ordering on $\mathbb{N} \cup \{\infty\}$.

The reason we care about this is the following theorem:

Theorem. Suppose that I is a closed interval and f is any continuous function from I to itself. Then, if f has a n-periodic point, it has a m-periodic point for any $n \lhd m$ (under the Sharkovsky ordering.)

In other words, it's not simply true that period 3 implies every possible period! We have many other strange consequences, like the following:

- Period 5 implies every period except for possibly 3.
- Period 7 implies every period except for possibly 3 and 5.
- Period 6 implies all of the even periods.
- Any function with finitely many periodic points must only have points with periods equal to powers of 2.

How would we prove such a thing? Well: given that we already have a functioning proof that period 3 implies chaos, perhaps we can adapt it to prove this stronger claim!

If you break down our proof from earlier, it effectively has three main parts:

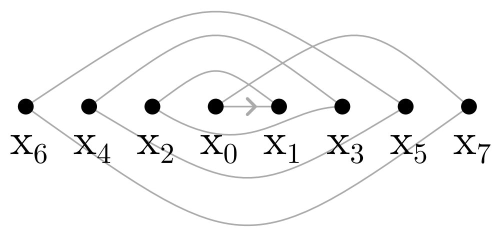
- First, we recognized that any triple $x_0 < x_1 < x_2$ corresponding to a point of period 3 has one of two orbits $x_1 \to x_0 \to x_2$ or $x_1 \to x_2 \to x_0$.
- Then, we translated these orbits into intervals $[x_0, x_1], [x_1, x_2], [x_0, x_2]$, and noted that we have either
  - $[x_0, x_1]^* \leftrightarrows [x_1, x_2]^* \circlearrowleft$ or
  - $[x_1, x_2]^* \leftrightarrows [x_0, x_1]^* \circlearrowleft$.
- Finally, we came up with a clever way to apply the Itinerary Lemma to secuences made out of these two intervals, that let us create points with any period.

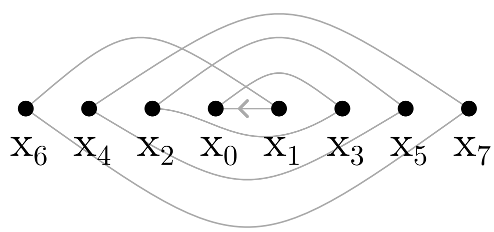
Therefore, if we want to modify this proof, we should start by examining the first of these parts! We do this here:

Lemma. Suppose that f is a continuous function on some interval I with the following two properties:

- There is some n such that f has a point, x_0, of period $2n+1$.
- For all $1 \leqslant m < n$, f has no points of period $2m+1$.

Then the orbit of x_0 must look like one of the following:





In other words: if we define $x_i = f^i(x_0)$, we have either

$$x_{2n} < x_{2n-2} < \ldots < x_4 < x_2 < x_0 < x_1 < x_3 < \ldots < x_{2n-1}, \text{ or}$$
$$x_{2n-1} < x_{2n-3} < \ldots < x_3 < x_1 < x_0 < x_2 < x_4 < \ldots < x_{2n}.$$

Before we start our proof, let's try to understand what we're even attempting to do here.

First: to make our notation easier, reorder the $2n+1$ points in the orbit of x_0 as the sequence

$$z_1 < z_2 < \ldots < z_{2n} < z_{2n+1}.$$

Now: let's look at what we're trying to prove. In both of the pictures that we're trying to prove must hold, we have this sort of "spiraling-out" relation, where

- an initially small interval $[x_0, x_1]$ becomes after one application of f the slightly larger interval $[x_2, x_1]$,
- which after another application of f becomes the larger interval $[x_2, x_4]$,
- which after yet another application of f becomes the larger interval $[x_3, x_4]$,

- ...
- which at our last stage becomes our entire collection of points!

Furthermore, this "spiraling-out" relation looks like pretty much enforces the structure we're claiming must exist: if there is some seed interval $[z_m, z_{m+1}]$ such that

- Repeatedly applying f to this interval expands it by precisely one z_i at each application, and
- the"side" this z_i shows up on alternates from left to right,

we've got the structure that we want! (Sketch out anything satisfying the two properties above to see why this is sufficient.) This suggests that if we want to prove our claim, we should attempt to look for this sort of interval structure in our z_i's.

In tomorrow's lecture, we'll do precisely this!