Mathcamp Crash Course	Instructor: Padraic Bartlett
Lecture 3: Equivalence Relations	
Week 1	*Mathcamp 2014*

In our last three talks of this class, we shift the focus of our talks from proof techniques to proof "concepts" that come up all the time in mathematics. Today's concepts are the ideas of **sets** and **equivalence relations**:

1 Sets

A **set**, for the purposes of this lecture, is just some collection of objects[1]. We usually denote a set by listing its elements in between a pair of curly braces $\{\}$. For example, $\{1, 2, 3\}$ is the set containing the numbers 1, 2 and 3, while $\{1, 2, salmon\}$ contains the numbers 1 and 2, along with a salmon. We will often give these sets names, and write things like $A = \{1, 2, salmon\}$ so that we can refer to the set containing 1, 2, and a salmon without having to write out all of the things in that set every time.

We call the objects that make up a set the **elements** or **members** of that set. If we want to say that a given object is in a set, we express this with the symbol $\in$, pronounced "in." For example, we write things like $2 \in A$ to express the notion that 2 is an element of the set A we defined earlier.

Sometimes, we will want to define a set without writing down all of the elements in the set. In these cases, we can instead define a set by writing down a **rule** that determines whether or not a given number is a member of that set.

For example, we can't define the set of natural numbers $\mathbb{N}$ by writing down every element in $\mathbb{N}$: there are infinitely many elements we'd have to write! Instead, what we can do is give a **rule** that determines whether a number is in $\mathbb{N}$: namely, a number is in $\mathbb{N}$ if it is a whole number that is nonnegative. Formally, we write this as

$$\mathbb{N} = \{a | a \in \mathbb{N} \text{ exactly whenever } n \text{ is a nonnegative whole number.}\}$$

The rule that we're proposing for our set — "a is a natural number precisely whenever a is a nonnegative and whole number" — goes on the right of the vertical bar $|$. On the left of the bar, we put the variable a, so that when we're reading our rule we know what letter corresponds to the elements of our set. Strictly speaking, the part on the left of this vertical bar isn't necessary for understanding what's going on in this notation; the rule we've written tells us everything we're looking for! However, it **makes our life easier** to have a reminder before we read our rule that the variable we care about is a. This is a thing you'll run into a lot in future math/physics classes: it's often as important to make your answers and work **easily understood** as it is to make it correct. Eventually, the ideas we start grappling with in the sciences are at the limits of human comprehension; a breakthrough in

[1] If you go further off into mathematics and the field of set theory, it turns out that this definition breaks down in some fairly strange and unexpected ways: you can construct sets that wind up doing remarkably awful things if you think of them as just arbitrary collections! This isn't the point of this lecture, but if you're interested I recommend checking out the wikipedia article on Russell's paradox for more information.

notation that simplifies the concepts at hand can sometimes be more valuable than a dozen new discoveries!

It is possible to write a set in many different ways. For example, we could write $\mathbb{N}$ as the set

$$\mathbb{N} = \{a | a \text{ is either equal to } 0, \text{ or there is some other number } b \in \mathbb{N} \text{ such that } a = b + 1.\}$$

This definition is nice because it doesn't rely on a reader already knowing what "whole" numbers or "nonnegative" numbers are; instead, it simply defines a natural number as something that is either 0, or something you can get by adding 1 to another natural number. So 1 is a natural number, because you can get 1 by adding 1 to 0. With this observation, we can see that 2 is a natural number, because you can get 2 by adding 1 to 1, and we know that 1 is a natural number. Then we can see that 3 is a natural number, because we can get 3 by adding 1 to 2, which we just showed was a natural number ... and so on and so forth.

Some textbooks will often just write some of the elements in a set, instead of giving a rule that describes the elements in the set, as a way of describing the set in a situation where the set is already well-understood. For example, many textbooks will write

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$
$$\mathbb{Z} = \{\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots\}$$

to describe the natural numbers and integers, respectively.

2 Equivalence Relations

Definition. Take any set S. A **relation** R on this set S is a map that takes in ordered pairs of elements of S, and outputs either true or false for each ordered pair.

You know many examples of relations:

- Equality ($=$), on any set you want, is a relation; it says that $x = x$ is true for any x, and that $x = y$ is false whenever x and y are not the same objects from our set.
- "Mod n" ($\equiv \mod n$) is a relation on the integers: we say that $x \equiv y \mod n$ is true whenever $x - y$ is a multiple of n, and say that it is false otherwise.
- "Less than" ($<$) is a relation on many sets, for example the real numbers; we say that $x < y$ is true whenever x is a smaller number than y (i.e. when $y - x$ is positive,) and say that it is false otherwise.
- "Beats" is a relation on the three symbols (rock, paper, scissors) in the game Rock-Paper-Scissors. It says that the three statements "Rock beats scissors," "Scissors beats paper," and "Paper beats rock" are all true, and that all of the other pairings of these symbols are false.

In this class, we will study a specific class of particularly nice relations, called **equivalence relations**:

Definition. A relation R on a set S is called an **equivalence relation** if it satisfies the following three properties:

- **Reflexivity**: for any $x \in S$, xRx.
- **Symmetry**: for any $x, y \in S$, if xRy, then yRx.
- **Transitivity**: for any $x, y, z \in S$, if xRy and yRz, then xRz.

It is not hard to classify our example relations above into which are and are not equivalence relations:

- Equality $(=)$ is an equivalence relations on any set you define it on – it trivially satisfies our three properties of reflexivity, symmetry and transitivity.
- "Mod n" ($\equiv \mod n$) is an equivalence relation on the integers. This is not hard to check:
  - **Reflexivity**: for any $x \in \mathbb{Z}$, $x - x = 0$ is a multiple of n; therefore $x \equiv x \mod n$.
  - **Symmetry**: for any $x, y \in S$, if $x \equiv y \mod n$, then $x - y$ is a multiple of n; consequently $y - x$ is also a multiple of n, and thus $y \equiv x \mod n$.
  - **Transitivity**: for any $x, y, z \in S$, if $x \equiv y \mod n$ and $y \equiv z \mod n$, then $x - y$, $y - z$ are all multiples of n; therefore $(x - y) + (y - z) = x - y + y - z = x - z$ is also a multiple of n, and thus $x \equiv z \mod n$.
- "Less than" $(<)$ is not an equivalence relation on the real numbers, as it breaks reflexivity: $x \not< x$, for any $x \in \mathbb{R}$.
- "Beats" is not an equivalence relation on the three symbols (rock, paper, scissors) in the game Rock-Paper-Scissors, as it breaks symmetry: "Paper beats rock" is true, while "Rock beats paper" is false.

Equivalence relations are remarkably useful because they allow us to work with the concept of equivalence classes:

Definition. Take any set S with an equivalence relation R. For any element $x \in S$, we can define the **equivalence class** corresponding to x as the set

$$\{s \in S \mid sRx\}$$

Again, you have worked with lots of equivalence classes before. For $\mod 3$ arithmetic on the integers, for example, there are three possible equivalence classes for an integer to belong to:

$$\{\ldots -6, -3, 0, 3, 6\ldots\}$$
$$\{\ldots -5, -2, 1, 4, 7\ldots\}$$
$$\{\ldots -4, -1, 2, 5, 8\ldots\}$$

Every element corresponds to one of these three classes.

The concept of equivalence classes is useful largely because of the following observation:

Observation. Take any set S with an equivalence relation R. On one hand, every element x is in some equivalence class generated by taking all of the elements equivalent to x, which is nonempty by reflexivity. On the other hand, any two equivalence classes must either be completely disjoint or equal, by symmetry and transitivity: if the sets $\{s \in S \mid sRx\}$ and $\{s' \in S \mid s'Ry\}$ have one element t in common, then tRx and tRy implies, by symmetry and transitivity, that xRy; therefore, by transitivity, any element in one of these equivalence relations must be in the other as well.

Consequently, these equivalence classes **partition** the set S: i.e. if we take the collection of all distinct equivalence classes, every element of S is in exactly one such set.

3 The Rational Numbers

One particularly useful use of the concept of equivalence classes is the definition of the rational numbers! In particular, ask yourself: what is the set of the rational numbers?

Most people will quickly say something equivalent to the following:

$$\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

The issue with this as a set is that it has **lots** of different entries for numbers that we usually think are not different objects! I.e. the set above contains

$$\frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots,$$

all of which we think are the same number! People usually then go back and change our definition above to the following:

$$\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b > 0, GCD(a, b) = 1 \right\}.$$

This fixes our issue from earlier: we no longer have "duplicated" numbers running around. However, it has other issues: suppose that you wanted to define addition on this set! Naively, you might hope that the following definition would work:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

However, for many fractions, the output of this operation is not an element of our new set!

$$\frac{2}{5} + \frac{8}{5} = \frac{40 + 10}{25} = \frac{50}{25} \notin \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b > 0, GCD(a, b) = 1 \right\}.$$

These difficulties that we're running into with the rational numbers come from the fact that, practically speaking, **they aren't a set** in most contexts that we work with them! Rather, they are a **set with an equivalence relation**:

1. The underlying set for the rational numbers: $\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}$.

2. The equivalence relation: we say that $\frac{a}{b} = \frac{c}{d}$ if there are a pair of integers k, l such that $ka = lc$ and $kb = ld$.

3. A **rational number** is any equivalence class of our set above under the above equivalence relation. This is the idea we have when we think of

$$\frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots$$

as all representing the "same number" 1/2: we're identifying 1/2 with its equivalence class!

4. In this setting, we define addition, multiplication, and all of our other properties just how we would normally: i.e. we define

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

where the only wrinkle is that by each of $\frac{a}{b}, \frac{c}{d}, \frac{ad+bc}{bd}$ we actually mean "take any element equivalent to these fractions," and by equality above we actually mean our equivalence relation.

Actually proving this is an equivalence relation is a task we leave for the homework! Do it if you're interested.