Mathcamp Crash Course	Instructor: Padraic Bartlett

Homework 3

Week 1	*Mathcamp 2014*

Homework instructions: some of the problems below are labeled with the tag $(*)$. $(*)$ denotes that the problem in question is fairly fundamental to the topics we're studying, and is something that you should probably be able to solve. If you get stuck on any problem, or see a typo, find me! I can offer tons of hints and corrections.

This class is homework-required! What this means is that I'm expecting you to try every problem, to solve all of the $(*)$ ones, and to solve at least some of the other problems.

HW is due at the start of class every day! I'll try to look over solutions in between classes, and come up with comments for you in time for TAU. Relatedly: come find me at TAU each day to get your HW, and to talk about how you're doing in the class!

1. [$(*)$] We defined the concept of a **relation** $R : S^2 \to \{T, F\}$ on a set S in class on Monday, and discussed three properties that relations can have:

   - **Reflexivity**: for any $x \in S$, xRx.
   - **Symmetry**: for any $x, y \in S$, if xRy, then yRx.
   - **Transitivity**: for any $x, y, z \in S$, if xRy and yRz, then xRz.

   Any relation can satisfy or not satisfy each of the above properties, for a total of eight possible combinations $(2 \cdot 2 \cdot 2)$.

   Come up with eight relations, one for each of the eight possible truth triples for reflexivity, symmetry, and transitivity. All of these relations should be on sets that are not numbers and that are definable outside of mathematics (i.e. "x is heavier than y," on the set of planets in our solar system; "x has dunked on y," on the set of players in the NBA; etc.)

2. [$(*)$] If you successfully completed problem 1 above, you found a relation that is symmetric and transitive, but not reflexive. This means that the following proof, that any symmetric and transitive relation must be reflexive, is flawed in some way:

   Claim 1. *Suppose that R is a symmetric and transitive relation on a set S. Then R is also reflexive.*

   Proof. Take any two elements $x, y \in S$. We know by symmetry that xRy implies yRx. Similarly, by transitivity, we know that xRy and yRx implies xRx. Therefore, for any $xinS$, we have xRx; i.e. the relation R is reflexive. $\square$

   Find the flaw in this proof.

3. Suppose that X, Y are a pair of statements that depend on a number of variables a, b, c, d, e, f, g. Consider the statement

$$\neg\Big(\forall a\ \exists b\ \forall c\ \exists d\ \forall f\ \exists g\ ((\neg X) \Rightarrow Y)\Big).$$

Write this statement without using the symbol $\neg$, the word "not", or any other methods of negation.

4. Suppose you have n distinct beads, which you want to use to make a necklace (which is just a circular loop on which our beads are placed in some order.) Suppose we call two necklaces **indistinguishable** if we can rotate or flip one necklace so that it is one of the other necklaces. Prove that indistinguishability is an equivalence relation. For any n, how many distinct equivalence classes exist on the set of n-bead necklaces?

5. There are two possible equivalence relations on the two-element set $\{1, 2\}$:

   - The relation $\sim_A$ that says that $1 \sim_A 1$ and $2 \sim_A 2$ are true, and that $1 \sim_A 2$ and $2 \sim_A 1$ are false.
   - The relation $\sim_A$ that says that $1 \sim_A 1$, $2 \sim_A 1$, $1 \sim_A 2$, and $2 \sim_A 2$ are all true.

   Find all of the possible equivalence classes on the set $\{1, 2, 3, 4\}$.

6. In class, we came up with a relation on the set of all fractions of the form $\frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0$ that we said was an equivalence relation (and in particular that created the rational numbers.) Prove that this relation is in fact an equivalence relation.