Mathcamp Crash Course	Instructor: Padraic Bartlett

Homework 2

Week 1 | *Mathcamp 2014*

Homework instructions: some of the problems below are labeled with the tag $(*)$. $(*)$ denotes that the problem in question is fairly fundamental to the topics we're studying, and is something that you should probably be able to solve. If you get stuck on any problem, or see a typo, find me! I can offer tons of hints and corrections.

This class is homework-required! What this means is that I'm expecting you to try every problem, to solve all of the $(*)$ ones, and to solve at least some of the other problems.

HW is due at the start of class every day! I'll try to look over solutions in between classes, and come up with comments for you in time for TAU. Relatedly: come find me at TAU each day to get your HW, and to talk about how you're doing in the class!

1. [$(*)$] Let x be an integer. Show that x is a multiple of five if and only if x^2 is a multiple of five.

2. [$(*)$] Find the flaw in the following proof:

   Theorem 1. *All ponies are the same color.*

   Proof. We proceed by induction. Specifically, let $P(n)$ be the claim "In any collection of n ponies, all of these ponies are the same color."

   Base case: we want to prove $P(1)$. But $P(1)$ is trivially true; in any collection made of one pony, all of the ponies in that set are the same color.

   Inductive case: we want to prove that $P(n) \Rightarrow P(n+1)$. In other words, we want to prove that whenever $P(n)$ is true, $P(n+1)$ is also true. To do this: assume that $P(n)$ is true, i.e. that in any set of n ponies, all of those ponies are the same color. With this assumption, we want to prove that $P(n+1)$ is true: i.e. that in any set of $n+1$ ponies, all of these ponies are also the same color.

   To do this: take any set of $n+1$ ponies, and write them as the set $\{p_1, \ldots p_{n+1}\}$. Break this set up into two subsets of size n: the subset $\{p_1, \ldots p_n\}$ and the subset $\{p_2, \ldots p_{n+1}\}$. These are both sets of size n: by our inductive hypothesis, they are both the same color. But these sets share the ponies $p_2, \ldots p_n$ in common! Therefore, whatever color our first set $\{p_1, \ldots p_n\}$ is **must be** the same color as the second set $\{p_2, \ldots p_{n+1}\}$, because they overlap! Therefore, all of our $n+1$ ponies are the same color, and we've proven that $P(n+1)$ is true (given our assumption $P(n)$.)

   So: we've proven that $P(1)$ is true, and that $P(n) \Rightarrow P(n+1)$. Therefore, by induction, we've proven that our claim $P(n)$ is true for all n; if we let n be the total number of ponies in existence, this proves our claim. □

3. [$(*)$] Prove that there are infinitely many prime numbers.

4. [(∗)] Find the flaw in the following proof:

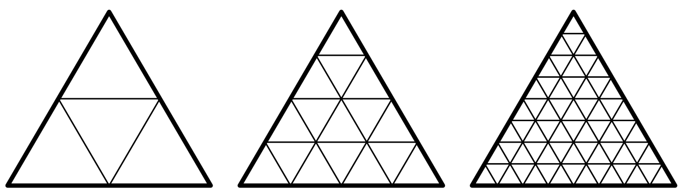
   Theorem. On a certain island, there are $n \geq 2$ cities, some of which are connected by roads. If each city is connected by a road to at least one other city, then you can travel from any city to any other city along the roads.

   Proof. We proceed by induction on n.

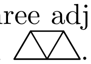
   Base case: the claim is clearly true for $n = 1$.

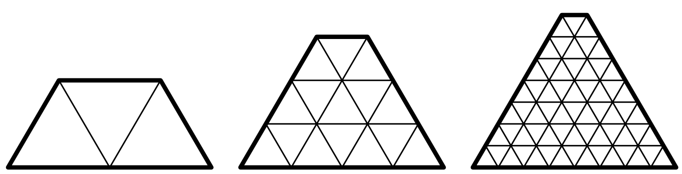
   Inductive step: suppose the claim is true for an island with $n = k$ cities. To prove that it's also true for $n = k + 1$, we add another city to this island. This new city is connected by a road to at least one of the old cities, from which you can get to any other old city by the inductive hypothesis. Thus you can travel from the new city to any other city, as well as between any two of the old cities. This proves that the claim holds for $n = k + 1$, so by induction it holds for all n. $\square$

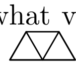
5. Take any $n \in \mathbb{N}$. Form the equilateral triangle $T(n)$ with side length 2^n, and subdivide this triangle into equilateral triangles of side length 1 (as drawn below.)



   (a) How many of these smaller triangles are there in $T(n)$? Explain why this is true.

   (b) A **triangular triomino**, for the purposes of this problem set, is a tile consisting of three adjacent equilateral triangles of side length 1: namely, it's a tile of the form . Suppose you take any n, construct $T(n)$, and remove one of the corner triangles from $T(n)$.



   For what values of n can you exactly cover all of the remaining triangles using these  three-triangle tiles (along with their flips and rotations?)

6. The game of **generalized n-tic-tac-toe** is played as follows: on a $n \times n$ grid, two players X and O take turns placing their respective symbols x, o into cells of the grid. No cell can be repeated. The game ends whenever any player gets n consecutive copies of their symbol on the same row /column / diagonal, or when the grid is completely filled in without any player having any such n consecutive symbols. (Normal tic-tac-toe is where $n = 3$.)

Prove that there is no strategy in generalized tic-tac-toe where the **second player** to move is guaranteed to win.

7. Prove that every fourth Fibonacci number is a multiple of 3. (Hint: show that $f_{4k+4} = 5f_{4k} + 3f_{4k-1}$, for any k.)