Electrical Networks and Graphs — Professor: Padraic Bartlett

Homework + Lecture 5: Random Walks on $\mathbb{Z}^3$

Week 2 — *Mathcamp 2014*

In our last class, we proved that random walks on $\mathbb{Z}^1$ and $\mathbb{Z}^2$ are **recurrent**: i.e. given enough time, a random walker on either graph will "eventually" return to the origin.

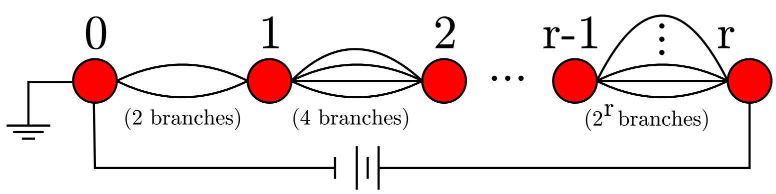
In today's talk, we will study $\mathbb{Z}^d$, for all $d \geq 3$! We start with $d = 3$, and prove the following claim:

Theorem 1. *The three-dimensional lattice graph $\mathbb{Z}^3$ is transient.*

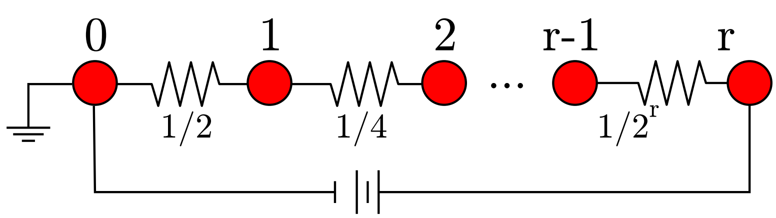
Proof. For $\mathbb{Z}^2$, the trick we used was to "short" a bunch of vertices together, and show that the resulting graph (which was simpler, even though its resistances were "lower") was recurrent. Here, in $\mathbb{Z}^3$, we're going to "cut" a number of resistors, and show that the resulting (simpler, higher-resistance) graph is transitive! (The normal proof of this theorem is much more difficult without these observations; it's only with this "shorting" and "cutting" that we can pull this off with such relative ease[1].)

In particular: lattices are **hard** to calculate resistances on. Let's try something simpler for a warm-up: a tree!

For example, let's consider the infinite binary tree graph T_2, where each edge has resistance 1, and we perform the standard trick of grounding everything at some cutoff distance r and put a potential of one volt at the root. Notice that (by symmetry) all of the nodes at any fixed distance k from the origin have the same potential: therefore, we can short them all together without changing the overall resistance of our circuit.



By using our earlier observations on resistors in parallel, we get that the above circuit is equivalent to the circuit below:



This has resistance $\sum 1/2^n = 1$.

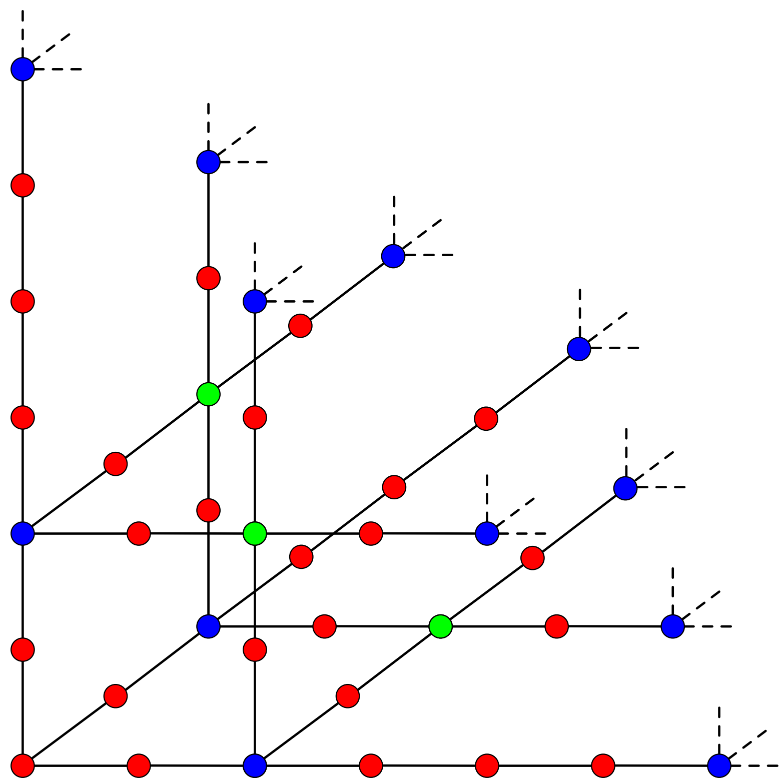
Naively, we might hope that we can just find a copy of T_2 in $\mathbb{Z}^3$, and be done with our argument. However, the number of nodes at distance n from the root of T_2 is 2^n, while the

[1] Insert your own "short-cut" pun here.

number of nodes that are distance $\leq n$ from the origin in $\mathbb{Z}^3$ is $O(n^2)$: so we're not going to be able to nicely fit a binary tree in $\mathbb{Z}^3$! What will we do?

Answer: we will be **clever**. Specifically, let's stay with the tree structure. Binary, however, may have been overkill: perhaps the sum $\sum 1/2^n$ converges far faster than we need! Instead, we could aim for a tree who splits often enough that we'll get *some* sort of convergent thing at the end of the day, but not so fast that we can't fit it in $\mathbb{Z}^3$. (This seems like a plausible goal: things like the sum $\sum \frac{1}{n^2}$ converge, so we certainly don't need as much branching as the binary tree T_2.)

To do this, consider the following kind of"tree:"



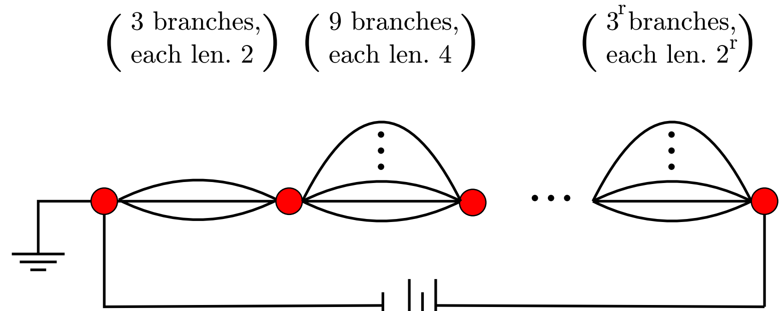
As currently drawn: not a tree. However, if you pretend that each of the green nodes are "doubled", by creating two vertices at each of those locations and passing only one branch through each node, it's a tree! Suppose for the moment that this picture is not lying to you: that the only overlapping parts of this tree are at the green vertices, and no branches or other such things overlap. Then, because the green nodes are at the same distances from the origin in the tree version of this graph, we know that they have the same voltage passing through them by symmetry — so there is no difference between the voltage/resistance/etc of the "tree" as drawn in our picture and the tree as realized by splitting the green nodes!

To give an explicit construction for the above picture: this tree is constructed by taking the positive octant of $\mathbb{Z}^3$ and starting from the origin. At the origin and each vertex with distance $\sum_{n=1}^{r} 2^n$ for every $r \geq 1$ (i.e. at the blue nodes,) our tree "branches" and creates three paths from these blue nodes: one branch that continues infinitely in the positive-x direction from that blue vertex, one that continues infinitely in the positive-y direction from that blue vertex, and one that continues infinitely in the positive-z direction from that blue vertex.

Notice that our tree only intersects at the "green" vertices in this picture, and specifically that these green vertices never coincide with one of these "blue" vertices. This is not hard to see: suppose that two tree branches managed to overlap at a blue vertex v that is distance $\sum_{n=1}^{r} 2^n$ from the origin. Then there must be two distinct blue vertices w_1, w_2 that we traveled from on distinct paths of length 2^r to get to v, both of which are distance $\sum_{n=1}^{r-1} 2^n$ from the origin. But this cannot happen: if we look at our two distinct paths of length 2^r, they are forming two sides of a square with side length 2^r in $\mathbb{Z}^3$ with v as one of its corners. Because the sum $\sum_{n=1}^{r-1} 2^n = 2^r - 2 < 2^r$, it is impossible for the two points w_1, w_2 to be distance $\sum_{n=1}^{r-1} 2^n$ from the origin and also be the two corners opposite v in this square.

Therefore our tree as drawn in $\mathbb{R}^3$ only overlaps at nodes that are not blue nodes, and therefore in particular only overlaps at vertices (i.e. it does not overlap on edges.) So, if we split it at each of these green nodes, we get an actual tree; moreover, because these vertices in the split tree all have the same voltages by symmetry, we can (again) see that there is no difference between the voltage/resistance/etc of the "tree" as drawn in our picture and the tree as realized by splitting the green nodes.

By identifying nodes of distance $\sum_{n=1}^{r} 2^n$ for every n from the origin, the graph on this tree restricted to the distances $\sum_{n=1}^{r} 2^n$ is equivalent to a circuit of the form



By applying our known results about resistors in series and parallel, we can see that the total resistance between any two nodes $n-1, n$ in the above circuit is

$$\frac{2^n}{3^n}.$$

Therefore, our tree at stage R has total resistance

$$\sum_{n=1}^{r} \frac{2^n}{3^n} = \frac{1-(2/3)^{r+1}}{1-(2/3)} - 1.$$

As r goes to infinity, this goes to 2; therefore, the current $i_A = v(A)/R_{\text{eff}} = 1/2$ at infinity is positive, and consequently the value $p_{\text{esc}} = i_A/C_A = \frac{1/2}{3} = 1/6$ is positive and nonzero. Therefore, by our earlier discussion, there is a nonzero chance of escape! In other words, our random walker may never return to the origin (and in fact, we've shown that they have at least a 1/6-th chance to do so!) $\square$

This gives us the following corollary, which is an excellent note to end our lecture on:

Corollary 2. *A lost drunkard will come home if and only if it cannot fly.*

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1. In class, we proved that p_{esc} on $\mathbb{Z}^3$ was at least $1/6$. The actual value of p_{esc} is actually $\sim .63$. By either finding a different tree, or somehow being clever in another way, improve the bound we came up with in class: show that $p_{\text{esc}} \geq 1/3$.

2. Show that p_{esc} on $\mathbb{Z}^4$ is at least $1/2$. Find bounds on p_{esc} for $\mathbb{Z}^d$, for all d. As d goes to infinity, does p_{esc} converge to 1?

3. (This problem assumes that you know some group theory.) Given a group A, the **Cayley graph** corresponding to A is defined as follows:

   Definition. Take any group A along with a generating set[2] element S. We define the **Cayley graph** $G_{A,S}$ associated to A as the following directed graph:

   - Vertices: the vertices of G_A are precisely the elements of A.
   - Edges: for two vertices x, y, create the oriented edge (x, y) if and only if there is some generator $s \in S$ such that $x \cdot s = y$. If this happens, we decorate the edge (x, y) with this generator s, so that we can keep track of how we have formed our connections.

   (a) Find a Cayley graph such that a random walker on that Cayley graph starting at the identity has a nontrivial chance of never returning to the identity. (Our walker ignores directions on edges.)

   (b) Find another Cayley graph, on infinitely many vertices, such that a random walker starting at the identity will eventually return to the identity with probability 1.

   (c) Take any group G that can be generated by a finite set of elements. Suppose that there is some finite set S of generators such that on the Cayley graph given by G and S, a random walker starting at the identity will eventually return to the identity. Show that under **any** finite set S of generators, a random walker starting at the identity will eventually return to the identity.

[2] A **generating set** for a group A is simply some collection S of elements of A, such that we can create **any** element of A by combining elements from our generating set and/or taking inverses. For example, the integers have $\{1\}$ as one possible generating set; however, $\{2, 3\}$ is another valid generating set, as is $\{1, 2, 3\}$ (redundancy is OK!)