Electrical Networks and Graphs	Professor: Padraic Bartlett

Homework + Lecture 2: Effective Resistance

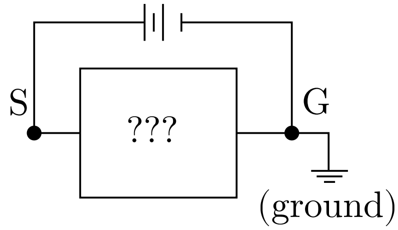
Week 2 | *Mathcamp 2014*

Most of this talk was centered around finishing up the notes from day 1! At the end, though, we introduced one useful concept that wasn't on those notes: the idea of **effective resistance**.

1 Circuits as Black Boxes

Suppose that we have a circuit with two points S, G, where we've grounded G and have a voltage of $1v$ established at S. If you have done this, then there is some amount of current flowing out of S. Denote this quantity as i_S, and note that i_S is given by the sum $\sum_{x \in N(S)} i_{Sx}$. Note that by Kirchoff's laws, the quantity of current that flows out of S is the same as the quantity of current that flows into G, because the sum of the currents through every vertex not S, B is equal to 0, and therefore whatever flows out of S must eventually flow into G.

Now, imagine simply covering up all of the connections and other bits between S and G with some sort of big black box. If we do this, then our circuit just looks like the following:



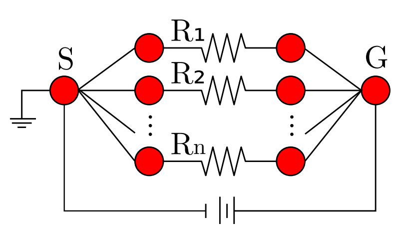
In this sense, we can simply "abstract" the rest of our circuit as some particularly large and bulky resistor, with effective resistance (which we denote R_{eff}) defined by Ohm's law:

$$\frac{V(S) - V(G)}{i_A} = R_{\text{eff}}$$

Similarly, we can define $C_{\text{eff}} = 1/R_{\text{eff}}$.

This is a useful concept! In particular, this lets us "simplify" electrical networks (and their corresponding random walks) by replacing large chunks of complicated circuits with much simpler circuits.

For example, on the last HW, I asked you to look at the following structure:

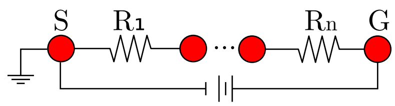


In this picture, the effective resistance of the circuit above is the reciprocal of the sum of the reciprocals of the resistors:

$$\frac{1}{R_{\text{eff}}} = \sum_{i=1}^{n} \frac{1}{R_i}.$$

Alternately, you can think of this claim as the statement that the "effective conductance" of the circuit is the sum of the conductances of the circuit.

Similarly, suppose we have a circuit made of resistors linked in series, as depicted below:



Then the effective resistance of the circuit above is the sum of the resistors:

$$R_{\text{eff}} = \sum_{i=1}^{n} R_i.$$

So: if you have a circuit with a chain of resistors in series, you can replace that chain with one single resistor with resistance R_{eff}. This process doesn't change the currents in our circuit, by construction, nor the resistances on anything we're not replacing with R_{eff}. As a consequence, the voltages also don't change (as they're defined by Ohm's law in terms of resistance and current!)

Consequently, the corresponding random walk can be simplified in the same way! This is really useful, and makes our lives great.

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Homework 2

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We start by mentioning that today's lecture will probably help a lot with understanding yesterday's problem set:

0. If you couldn't solve the problems from yesterday because we didn't have the tools to do so yet, try them now!

New problems start here.

So: in class, we connected a random walk on a graph G with edge weights w_{xy} to a circuit on the same graph with conductances $C_{xy} = w_{xy}$, via the following correspondence: $p(x) = v(x)$, if $v(x)$ is the voltage function and $p(x)$ is the probability that a random walker starting at x makes it to the source before the ground.

This raises a fairly natural question: if voltage has a nice interpretation in terms of random walks, do other properties of circuits have nice interpretations? Specifically: how about current?

The answer is yes! We prove this via the following two exercises:

1. Prove the following lemma:

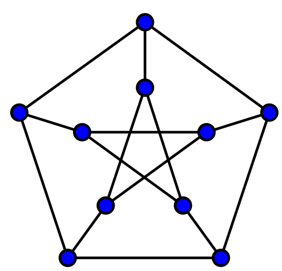
   Lemma 1. *Take a graph H with source vertex S, ground vertex G, and edge weights $w_{xy} \in \mathbb{R}^+$. Suppose that we model a random walk on this graph as follows: start at S, and at each time step randomly (using our edge weighting, as discussed in the notes) pick a neighbor of the vertex that it is currently at and wanders to that vertex. If it ever reaches G it stops; otherwise, it continues. (This is unlike in our previous interpretation, where the walker also stopped if it arrived at the source vertex S.) Let $u(x)$ denote the average number of times that a walker will wander through the vertex x before reaching G in this model.*

   Consider the corresponding circuit to this weighted graph, where all edges have conductance corresponding to our weights, G is grounded, and a potential of $\frac{u(S)}{C_S}$ volts is established between S and G. Then we have the following relation between $v(x)$, and $u(x)$, valid at every vertex x of our graph:

$$v(x) = \frac{u(x)}{C_x}.$$

2. Take a circuit of the form described in problem 1, with corresponding random walk problem. Show that for any edge $\{x, y\}$, i_{xy} is equal to the expected number of times that a random walker goes from x to y along this edge, minus the expected number of times that a random walker goes from y to x along the same edge.

This last problem is just here because I'm curious about the answer. Consider the Petersen graph:



3. Turn the Petersen graph into a circuit as follows:

   - Pick any two nonadjacent vertices S, G in this graph. Set one to be the source and the other to be grounded. Define the voltage at the source to be 1, the voltage at ground to be 0, and the resistance of every edge of this graph to be 1.

     What is the effective resistance of this circuit?

4. Did it matter what vertices you chose for S, G? Or is the answer the same for any two such vertices?