Algebraic GT	Instructor: Padraic Bartlett

Lecture 2: Schreier Diagrams

Week 5	*Mathcamp 2014*

This class's lecture continues last's class's discussion of the interplay between groups and graphs. In specific, we define the **Schreier diagram** in these notes, calculate some examples, and (if there is time) look at some applications of these techniques!

1 Schreier graphs

Definition. Take a group G, a subgroup H of G, and some collection of elements S that (along with the elements in H) generate G. We create the **Schreier diagram** corresponding to this collection of information as follows:

- Vertices: the various right cosets of H in G.
- Edges: connect two cosets K, L with an edge if and only if there is some element $s \in S$ such that $Ks = L$.

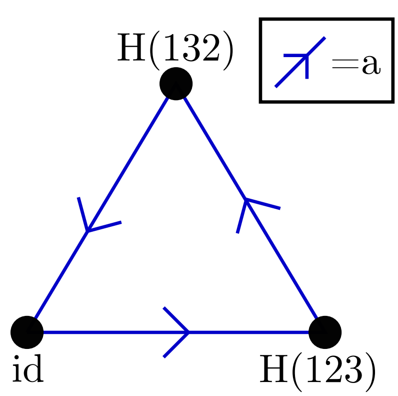
In this sense, a Cayley graph is simply a Schreier diagram where we set $H = \{id\}$.

We consider a pair of examples:

Examples. Let's take $G = S_3$ as before, with the subgroup $H = \{id, (12)\}$ and generating set $a = (123)$. This group has three possible cosets for H to bounce between:

$$\begin{aligned} H = H \cdot (12) =& \{id, (12)\}, \\ H \cdot (13) = H \cdot (132) =& \{(13), (132)\}, \\ H \cdot (23) = H \cdot (123) =& \{(23), (123)\}. \end{aligned}$$

This gives us a fairly simple Schreier diagram, if we use the fact that $a^2 = (132)$:



Examples. Consider the group $G = D_8$ = the collection of all symmetries of a square. We denote its eight elements, defined in last week's lecture notes, as the set

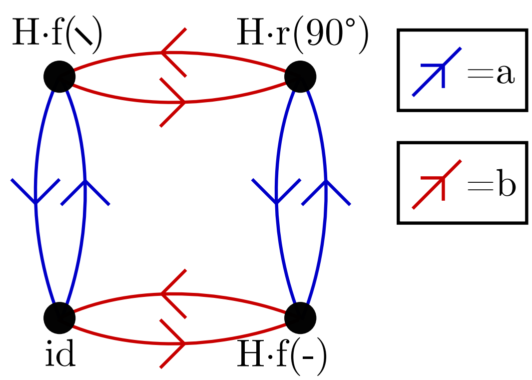
$$\{id, \text{rot}(90°), \text{rot}(180°), \text{rot}(270°), \text{flip}(\,|\,), \text{flip}(-), \text{flip}(\diagdown), \text{flip}(\diagup)\}.$$

By the "flip(line)" expressions, we mean the four symmetries of the square that consist of flipping the square over some axis, with the appropriate axis given in parentheses next to each flip.

Take the subgroup $H = \{id, \text{rot}(180°)\}$ along with the generators $S = \{a = \text{flip}(\diagdown), b = \text{flip}(-)\}$. Our subgroup has four possible cosets:

$$\begin{aligned}
H = H \cdot \text{rot}(180°) &= \{id, \text{rot}(180°)\},\\
H \cdot \text{rot}(90°) = H \cdot \text{rot}(270°) &= \{\text{rot}(90°), \text{rot}(270°)\},\\
H \cdot \text{flip}(\,|\,) = H \cdot \text{flip}(-) &= \{\text{flip}(\,|\,), \text{flip}(-)\},\\
H \cdot \text{flip}(\diagdown) = H \cdot \text{flip}(\diagup) &= \{\text{flip}(\diagdown), \text{flip}(\diagup)\}.
\end{aligned}$$

This gives us another fairly simple Schreier diagram:



The ease of the above two calculations indicates part of the reason why we might like Schreier diagrams: they are often easier to calculate than Cayley graphs. In exchange, however, we're only getting information about the cosets of H instead of the elements of our group — but if we only care about the elements of our group "up to" the elements H of our coset, this is still pretty great!

To illustrate a situation where working with the Schreier diagram is markedly easier than the Cayley graph, consider the following problem:

Problem. Consider the presented group

$$\langle a, b \mid a^2 = b^5 = (ba)^3 = id \rangle,$$

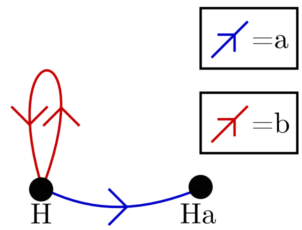
which has $< b \mid b^5 = id >= \{id, b, b^2, b^3, b^4\}$ as a subgroup. What is the Schreier diagram of this group with the generators $\{a, b\}$?

Answer. We use the same heuristics to find this Schreier graph that we used to find the Cayley graph for a presented group. We copy these heuristics from our earlier set of notes here:

0. Start by placing one vertex that corresponds to the "identity" coset H.
1. Take any vertex corresponding to a coset K that currently has a corresponding vertex in our graph. Because our graph is a Schreier graph, it must have one edge leaving that vertex for each generator in our generating set. Add edges and vertices to our graph so that this property holds.
2. If some word R_i is a word that is equal to the identity in our group, then in our graph the path corresponding to that word must be a **cycle**: this is because if this word is the identity, then multiplying any element in our group by that word should not change that element.

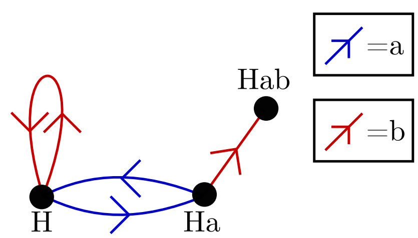
   Identify vertices only where absolutely necessary to insure that this property holds at every vertex. (This may be a bit trickier here, because we are dealing with cosets instead of group elements; consequently, it may take a bit of thought to determine what this condition is asking of us.)

We run this process here. We start with one vertex corresponding to the coset H:



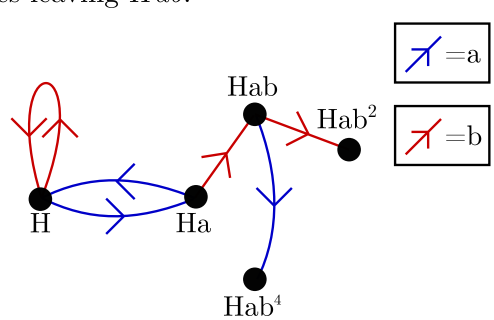
Note that because $Hb = H$, the b-edge leaving H returns to H itself, forming a loop. (This illustrates some of the slightly trickier aspects of working with cosets instead of groups. This, however, is the only time this will come up, which perhaps illustrates that cosets aren't so bad after all.)

We now take our one new vertex Ha and draw the two a, b-edges leaving Ha:

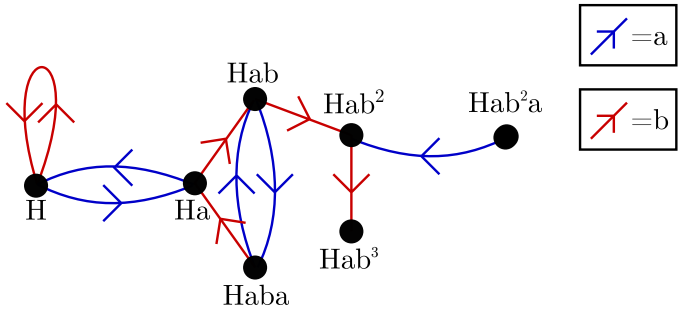


Here, we use the relation $a^2 = id$ to conclude that $Ha^2 = H$.

We now draw the edges leaving Hab:



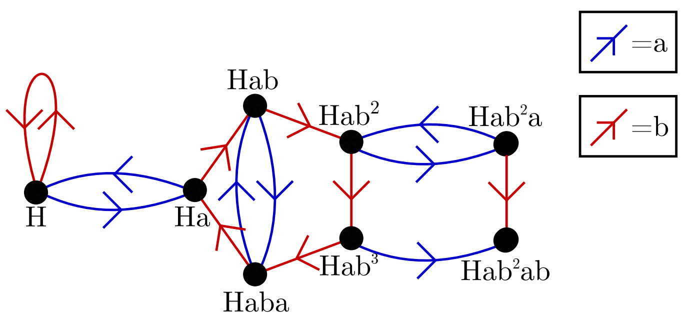
And repeat this process on $Hab^2, Haba$:



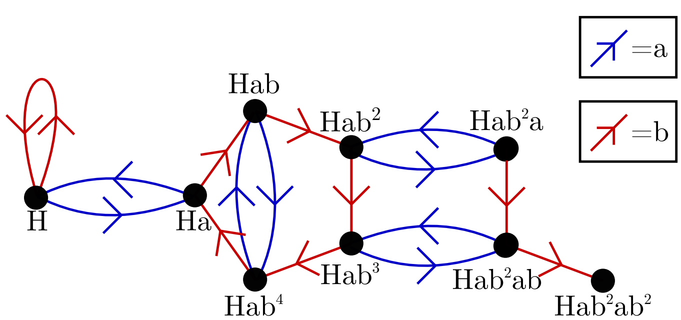
Notice here that the relation $a^2 = id$ means that the a-edge leaving $Haba$ returns to Hab; in general, this property will always insure that these a-edges come in pairs, and we will use this identification throughout the rest of this proof without calling it out.

More interestingly, note that $Habab = Ha$. This is because $bababa = id$ is equivalent to asking that the walk corresponding to $bababa$ starting at the origin returns to the origin. After the first four steps, we are at $Haba$; to return to H along an a-edge, we must go to Ha, which forces our connection.

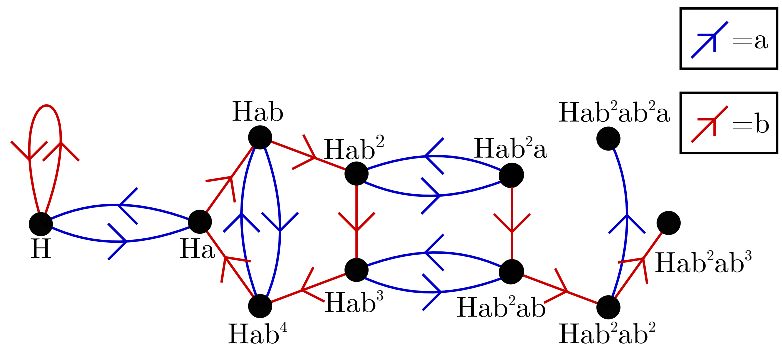
We draw more edges:



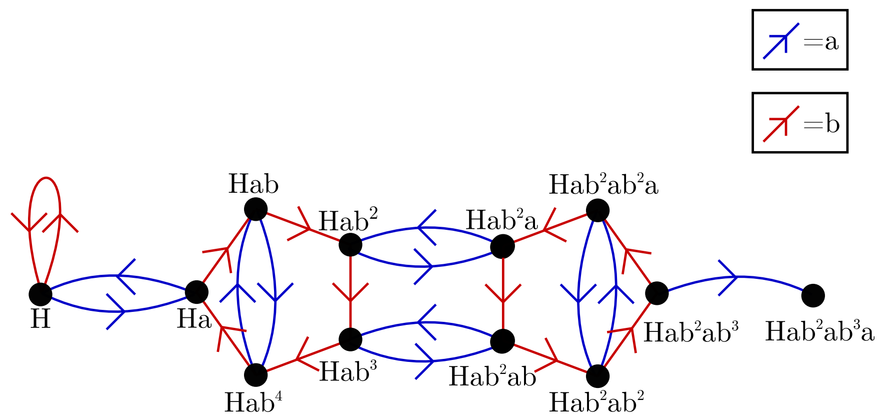
Notice that $Haba = Hab^4$; this is because if we start at $Haba$ and take the walk of length 5 given by the b-edges, we should return to ourselves. Also notice that $Hab^3a = Hab^2ab$; this is because the walk $bababa$ starting at Hab^3 must return to itself, and therefore that the a-edge leaving Hab^3 must go to whatever b-edge leaves Hab^2a.



Nothing nontrivial was identified above, so we continue our process:

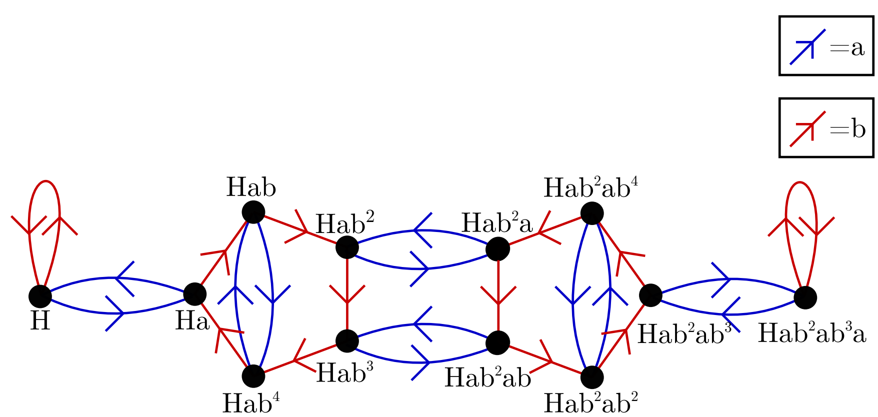


Still nothing. More edges!



Ok, now some interesting things have happened. Notice that we've identified Hab^2ab^2a with Hab^2ab^4; this is again because of the walk $bababa = id$, starting this time from the vertex Hab^2. In particular, because walking $baba$ from Hab^2 takes us to Hab^2ab^2a and walking ba more must return us to Hab^2, we know that our b-edge leaving Hab^2ab^2a must go to Hab^2a. Similarly, taking the walk b^5 starting from this Hab^2ab^2a vertex must return us to ourselves, forcing the b-edge leaving Hab^2ab^3 to go to Hab^2ab^2a.

We draw our last batch of edges:



Note that the b-edge leaving Hab^2ab^3a must return to itself, as the walk $bababa = id$ starting from the vertex Hab^2ab^3 forces the b-edge leaving Hab^2ab^3a to return to itself.

This gives us a ton of useful information about our group: it tells us that there are 60 elements (as we have 12 cosets, each containing 5 elements), and moreover it tells us how these cosets get moved around by a and b (in particular, looking at our graph tells us that b keeps two cosets constant and moves the other 10 around in two groups of 5.) For those of you who have done some group theory before, this actually is enough to tell us what this group is in its entirety (it's A_5, the alternating group on 5 elements!)

It turns out that adding a bit more information to our diagram can make them even more useful:

2 Decorated Schreier Diagrams

Definition. Given a Schreier diagram for a group G with subgroup H and generators S that we've labeled our edges with, we can **decorate** it! We do this as follows:

- Take all of the vertices of our Schreier diagram. Each vertex corresponds to a coset K. Pick some element $k \in K$, and use that element to **decorate** the vertex corresponding to that coset.

  Notice that if we have decorated a coset K with some element $k \in K$, then we can actually write $K = Hk$. So this decoration is a pretty natural one to use.

- Now, suppose that there is an a-edge going from one coset $K = Hk$ to another coset $L = Hl$. We decorate this edge with the group element α such that $ka = \alpha l$.

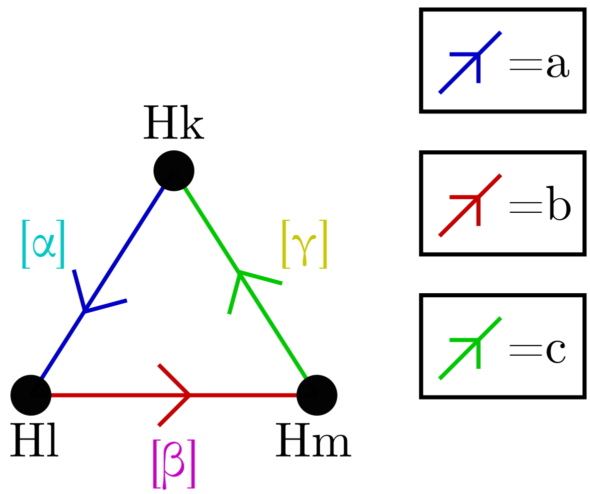
  Notice that because $L = Ka = Hka$, we can write $l = hka$ for some $h \in H$, and thus have $ka = \alpha hka \Rightarrow \alpha = h^{-1}$. In particular, this means that all of the edge decorations (1) exist, as we found a formula to find them, and (2) are all elements from our coset H.

Decorated Schreier diagrams satisfy a fairly interesting property:

Proposition. Take any Schreier diagram for a group G with subgroup H. Decorate it. Take any closed walk in our Schreier diagram that starts and ends at the H-vertex[1]. The product of the group elements used to label the edges of this closed walk, in the order given by our closed walk, is the same thing as the product of the group elements used to decorate our edges (in the order given by our closed walk.)

Proof. To illustrate the idea, let's take an arbitrary decorated three-vertex cycle starting from some coset Hk, where the edges are oriented as drawn below:

[1] In a directed walk, this is potentially ambiguous. For this talk, we mean any subgraph that when we forget the orientations of our edges, we get something that would be a closed walk in an unoriented graph.



A decorated three-cycle from within some Schreier graph. The vertices Hk, Hl, Hm are all decorated via their representatives k, l, m. There is an edge $Hk \to Hl$ given by the generator a, $Hl \to Hm$ given by the generator b, and $Hm \to Hk$ given by c; as well, these three edges are decorated by the labels α, β, γ.

Notice that because the a-edge $Hk \to Hm$ is decorated with an α, we have $ka = \alpha l$; similarly, because the b-edge $Hl \to Hm$ is decorated with β, we have $lb = \beta m$, and because the c-edge $Hm \to Hk$ is decorated with γ, we have $mc = \gamma k$.

Consequently, if we look at the product $kabc$, we have

$$kabc = \alpha lbc = \alpha\beta mc = \alpha\beta\gamma k.$$

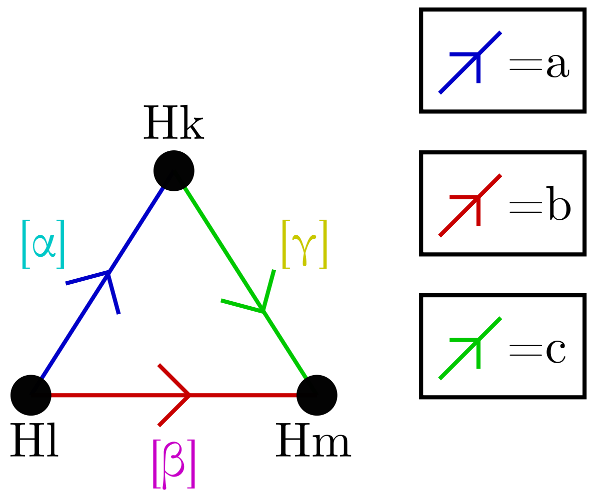
In particular, if $k = id$ — in other words, if $Hk = H$ — we have $abc = \alpha\beta\gamma$. In other words, the product of the "labels" on our cycle is the same thing as the product of the "decorations" on our cycle!

This proof generalizes to oriented cycles of length n by almost exactly the same proof: simply take any cycle with vertices decorated $k_1, \ldots k_n$, edges $k_i \to k_{i+1}$ labeled a_i and decorated α_i. Then by the exact same argument as above, we have

$$k_1 a_1 a_2 \ldots a_n = \alpha_1 k_2 a_2 \ldots a_n = \ldots \alpha_1 \alpha_2 \ldots \alpha_n k_1,$$

which gives us $a_1 a_2 \ldots a_n = \alpha_1 \alpha_2 \ldots \alpha_n$ in the case that the vertex corresponding to k_1 is the subgroup H (i.e. $k_1 = id$.)

We finally note that because the condition $ka = \alpha l$ is equivalent to the request $\alpha k = la^{-1}$, we can deal with the situation where edges are oriented in the "wrong" directions by simply replacing the a, α's with their inverses. For example, suppose we returned to our triangle from before, but messed with some of the orientations:



In this situation, we would use the relations

- $la = \alpha k \Rightarrow ka^{-1} = \alpha^{-1} l$,
- $lb = \beta m$,
- $kc = \gamma m \Rightarrow mc^{-1} = \gamma^{-1} k$

to perform the transformation

$$ka^{-1}bc^{-1} = \alpha^{-1}lbc^{-1} = \alpha^{-1}\beta mc^{-1} = \alpha^{-1}\beta\gamma^{-1}k.$$

Again, setting $k = 1$ gives us that the product of the labels of edges in our closed walk is the same as the product of the decorations of edges in our closed walk, provided that we interpret the "orientation" of each edge as telling us whether a group element is represented by itself or its inverse. $\square$

One convenient way to decorate a Schreier diagram is via the following process:

Proposition. Take any Schreier diagram for a group G with subgroup H. The following process induces a unique decoration of this diagram:

- Decorate the H-vertex with the element $id \in H$.
- Pick out some spanning tree[2] T in our graph. Decorate all of the edges in this spanning tree with the element $id \in H$.

Proof. This is not too hard to see. Look at any vertex K that is distance 1 from H, where we measure distance from the origin via our spanning tree: i.e. we are declaring that a vertex is distance n from H if there is a path of length n from H to that vertex in our spanning tree T. Because T is a spanning tree, this gives a well-defined distance function.

[2]Recall that a **spanning tree** of a graph G is a subgraph of G that (1) is a tree, and (2) contains every vertex in our graph. In this setting, where we are dealing with directed graphs, this notion might again be ambiguous; for this talk, we further define a tree as any subgraph that when we forget the orientations of our edges, we get something that would be a tree in an unoriented graph.

Suppose that the edge in our spanning tree connecting K to the origin is labeled a, and goes from $H \to K$. If we want H to be decorated as id and this a-edge to be labeled id, we are asking that the decoration of K is some element $k \in K$ such that $id \cdot a = k \cdot id$: i.e. that each of these vertices K has a unique decoration, given (in this particular case) by the edge-labeling that led to that coset.

The other case, where the edge goes from K to H, is similar; if we want H decorated as id and the a-edge $K \to H$ to be decorated id, then we must have K decorated with a k such that $ka = id \cdot 1 = id$, which again uniquely determines k. (This is like the orientations-corresponding-to-inverses relationship we saw in our earlier result.)

Now, suppose that we have decorated all of the vertices out to distance n, and want to decorate vertices at distance $n+1$. Take any K at distance $n+1$: because T is a spanning tree, there is some unique edge connecting a previously-decorated vertex L at distance n to our vertex K via an edge in T. Assume this edge is labeled with some element a, decorated by id, and that L is decorated with some element l.

Then, if the edge goes from $L \to K$, K must be decorated with an element k such that $la = id \cdot k$; similarly, if the edge goes from $K \to L$, K must be decorated with some k such that $ka = id \cdot l$. Notice that this uniquely defines K's labeling. Furthermore, notice that this labeling is conflict-free: because T is a tree, there is no way for us to have two conflicting claims as to what K's decoration should be.

This decorates all of the vertices in our graph. Now, take any edge $K \to L$ in our graph that we have not yet labeled (i.e. any edge not in the spanning tree.) Consider the closed walk formed by starting at H, walking to K along the unique path to K in our spanning tree, taking the edge $K \to L$, and walking back to H via the unique path back to H in our spanning tree. This is a closed walk; therefore, the product of the decorations of edges on this walk must be equal to the product of the labelings of edges on this walk!

But every edge in our walk is decorated by 1's, except for the $K \to L$ edge which we're trying to decorate. Therefore, this gives us a unique decoration of this edge, given by the labelings of the walks $H \to K$ and $L \to H$. So we've decorated our graph! $\square$

This method of decoration has an interesting consequence:

Theorem. Take any Schreier diagram for a group G with subgroup H, along with a generating set S for G. Decorate this diagram. Then the subgroup H is generated by the decorations of the edges in our graph.

Proof. Take any element $h \in H$. Because S generates G, we can write h as some product $s_1 \dots s_n$ of elements (possibly repeated and with inverses) from S. This corresponds to a walk on our Schreier graph: furthermore, because $s_1 \cdot \ldots \cdot s_n = h \in H$, this walk must start and end at H.

Decorate our Schreier diagram (say, using the decoration given above.) Now, the product of labels on this walk must be equal to the product of the decorations of the edges on this walk: in other words, we can write h as the product of some of the decorations of the edges in our graph! So any h can be written as the product of decorations in our graph.

Furthermore, by using walks that start at H and walk along edges in the spanning tree to get to any edge in our graph, walking on that edge, and then returning along our spanning tree edges, we can see that the decoration of any edge in our graph is an element in our subgraph. Therefore H is generated by these decorations, as claimed! $\square$

This theorem has the following very beautiful extension:

Corollary. Take any Schreier diagram for a group G with subgroup H, along with a generating set S for G. Decorate this diagram. Suppose that G has a presentation $\langle a_1, a_2, \ldots \mid R_1, R_2, \ldots \rangle$. Then the subgroup H has a remarkably nice presentation:

$$H = \langle d_1, d_2, \ldots \mid D_{1,1} D_{1,2} \ldots, D_{2,1}, D_{2,2}, \ldots \rangle,$$

where

- The generators $d_1, d_2, \ldots$ are all of the decorations of edges in our graph.
- The relations $D_1, D_2, \ldots$ are given by the following process: take any relation R_i from G. R_i corresponds to a labeled walk in G's Cayley graph, that starting from any vertex must return to that vertex. In other words, in our group, the product of the labelings on this walk's edges must be the identity.

  Now, we know that the product of the labels on this walk must be equal to the product of the decorations on this walk. In other words, a relation R_i on our generators can create several relations on the generators $d_1, d_2, \ldots$! Call these new relations $D_{i,1}, \ldots D_{i,n}$.

The main point of the above discussion is that all of the relations on H must come, in some sense, from pre-existing relations in G. (Think for a bit if this isn't clear; there is some nonobvious mathematics going on in this statement!)

If we consider the case where G is a free group (i.e. a group with no relations) we get the following result "for free:"

Corollary. Any subgroup of a free group is free.

For those of you who haven't done group theory before, this might not be very surprising, and seem like it should be a relatively "trivial" result. This is far from the truth; what we've presented here is the closest to a purely algebraic proof that is known, and is one of the simplest proofs I am aware of[3]!

[3] The other one I know goes through algebraic topology, and is similar in difficulty.