Proof Techniques | Instructor: Padraic Bartlett

Lecture 1: The Art and Technique of Proof

Week 1 | *Mathcamp 2012*

1 What is a Proof?

Every major field of study in academia, roughly speaking has a way of "showing" that something is true. In English/critical literature studies, if you wanted to argue that the concept of whiteness in Melville's Moby Dick was intrinsically tied up with mortality, you would write an essay that quoted Melville's epic story alongside some of of his other writings and perhaps some contemporary literature, and logically argue (using these quotations as "evidence") that your claim holds. Similarly, if you were a physicist and you wanted to demonstrate Bell's inequality, which roughly states that local realism and quantum mechanics are incompatible theories, you'd create an experiment under which these two theories necessarily predict different outcomes.

In mathematics, a **proof** is what we call an argument for showing that something is true. To define the concept of mathematical proof, then, it suffices to define the words "something," "argument" and "true." This may sound pedantic and perhaps silly, but consider the following cautionary examples of "failed" proofs:

Theorem. I am infallible.

Proof. First, recall the following two facts:

- All mentors are infallible.
- Paddy is a mentor.

Because Paddy is a mentor and all mentors are infallible, we can logically conclude that Paddy is infallible. □

The flaw here, obviously, is that one of the two things we claimed at the start of this proof – that all mentors are infallible – is clearly false.[1] Whenever we're making a proof, if we want to insure that we get something true at the end, we need to insure somehow that we never do something like the above; i.e. that we never accidentally assume false statements during the course of our proof. But this gets us back to one of the words we're trying to define: what does it mean for something to be true or false? Mathematically, we define these concepts recursively as follows:

Definition. A mathematical statement is **true** if and only if we have a mathematical proof for that statement.

[1] As anyone who's ever played frisbee with us can attest to. We fall over all the time.

But wait, you may protest: that's circular logic![2] If the only things we can use in mathematical proofs are true statements, and the only way we know if something is true is by finding a mathematical proof for it, we would seem to have no way of actually showing anything is true. To avoid this, we also introduce the concept of axioms:

Definition. We also have a small collection of statements, called **axioms**, which we assume to be true without proof.

For example, in **set theory**, we have the ZFC axioms of set theory, a collection of roughly ten rules that we can use to prove most of the major results in that branch of mathematics.

Typically, when we're proving a mathematical statement, we won't worry so much about whether the axioms we're working from are "naturally true" or not (what would this mean, anyways?); rather, we will often just state at the start of our proof what we're assuming to be true, and logically proceed from there. For example, the proof above **was** a completely logical proof of the claim "If all mentors are infallible, then Paddy is infallible:" if that first statement **was** true, then I would indeed be infallible.

This gives us a notion of "truth." Before we go much further, we should use this idea to give us an idea of what **statements** are:

Definition. A **statement** (or proposition, or claim) is just some object that is either true or false. For example, the following are statements:

- $P =$"Every even number greater than 2 can be expressed as the sum of six primes" is a statement; this one happens to be true (a result in number theory, proven in 1995 by the French mathematician Olivier Ramaré.)
- $Q =$ "Every even number can be expressed as the sum of two primes" is another statement; this one is false, as the number 2 cannot be expressed as the sum of two other primes (as there are no prime numbers smaller than 2.)
- $R =$ "Every even number greater than 2 can be expressed as the sum of two primes" is a third statement; this is Goldbach's conjecture, a famous open problem in number theory. It is either true or false, but mathematicians have not yet discovered which.

Often, we will work with mathematical statements that depend on a variable. For example, we can write

$$P(n) = \text{"A } n \times n \text{ checkerboard can be covered by nonoverlapping } 2 \times 1 \text{ dominoes;"}$$

this statement will be false for odd values of n, and true for even values of n (if you don't immediately see why, prove this!)

Definition. Given some statements, we will often want ways to combine them into new statements. The following list contains some of the most common combinations:

1. Given two mathematical statements P and Q, we will often want to form the mathematical statement "P and Q", denoted $P \wedge Q$. This denotes the mathematical statement that is true if and only if both P and Q hold, and is false otherwise.

[2] See footnote 2 for a definition of circular logic.

2. Given two statements P and Q, we can form the mathematical statement "P or Q", denoted $P \vee Q$. This denotes the mathematical statement that is false if and only if both P and Q are false, and is true otherwise.[3]

3. Given a statement P, we can formulate the mathematical statement "not-P," which we denote $\neg P$. This is the mathematical statement that is false whenever P is true, and true whenever P is false.

4. Given two statements P and Q, we can form the mathematical statement "P is equivalent to Q", denoted $P \Leftrightarrow Q$. This denotes the mathematical statement that is true when P and Q are either both true or both false, and false otherwise (i.e. when exactly one of P, Q are true, and the other is false.)

5. Given two statements P and Q, we can form the mathematical statement "P implies Q", denoted $P \Rightarrow Q$. This denotes the mathematical statement that is false if and only if P is true while Q is false; under any other situation, we consider $P \Rightarrow Q$ to be true. Notably, this means that if P is false, $P \Rightarrow Q$ is true no matter what Q does; this allows us to say that statements like "If I am a purple elephant, then six is an odd number" are true[4]. (Because the P part is false, it doesn't matter whether the Q part is complete nonsense or not; our implication is automatically true.)

Additionally, we will often use the shorthand $\forall$ "for all," $\exists$ "exists," $\in$ "in," and $\notin$ "not in," because we say these things all the time and it really simplifies statements.

We now understand the idea of truth, and how to work with and evaluate claims. This leaves us with one last object to clarify: the idea behind "argument." Consider the following cautionary example:

Theorem. All odd numbers are prime.

Proof. 3 is prime, 5 is prime, 7 is prime . . . seems to always hold. □

In this example, the issue is not that we introduced false statements: the numbers 3, 5 and 7 are all indeed prime. Rather, the problem is that the logic we used to link these facts to our conclusion — "if a statement holds for the first few examples we look at, it must be true in general" — is false. Discussing what it formally means for a piece of logic to be "valid" in a mathematical proof is a rather complicated thing to rigorously do (if you're interested, a course in first-order logic might be worthwhile); for our purposes, however, we won't worry about this too much. Specifically, you all already know pretty much what logical leaps are valid and which are not. For example, you know that the following logical constructions make sense:

[3] In mathematics, we almost always assume that our "or" is an inclusive-or: i.e. it is true when either P or Q is true, or even when **both** P and Q are true. In computer science, however, you will sometimes run into "exclusive-or," which is true when either P or Q is true, but is false when both are true. For your mathcamp classes, you're probably safe to assume that all "or" statements are inclusive-or, unless explicitly stated otherwise.

[4] Provided we are not purple elephants.

- If both of the statements A and B are true, then either one of the statements A or B are true: roughly speaking, this is like saying that if you have a dog and a cat, it is also true that you have a dog. In terms of the constructions above, this is saying that if $A \wedge B$ is true, then so is A (and similarly so is B.)

- If the statement A is true, and you know that whenever the statement A is true it forces the statement B to be true (in other words, you know that A implies B, or in symbols $A \Rightarrow B$), then you know that B must be true. Again, roughly speaking, this is like saying that knowing the two facts (it's raining) + (whenever it rains, it's wet outside) tells you that it's wet outside. In terms ofthe constructions above, this is saying that knowing that $A \Rightarrow B$ is true, along with A being true, tells you that B is true.

- If you know that $A \Rightarrow B$, and also that the statement B is false, then you know that there's no way that the statement A can also be true: i.e. that A is false as well. Again, to give an example, this is like stating that knowing (If there was not another season of My Little Pony, I would be sad) + (I am not sad) tells you that there will be another season of My Little Pony. In terms of the constructions from earlier, this is like saying that if $A \Rightarrow B$ is true and B is false, then A must also be false.

Conversely, you also know that the following arguments don't actually work for proving statements:

- Just because a property holds for the first few values you examine, doesn't mean that it's always true. A famous example is the Pólya conjecture, which fails only at $906,150,206$ but holds true for every number up to that value.

- Just because $A \Rightarrow B$, doesn't mean that $B \Rightarrow A$: a quick example is noting that just because I cheer at my television whenever Messi scores a goal, doesn't mean that Messi will score a goal if I cheer at my television. This is a special example of the idea that correlation does not imply causation: just because two events are related to each other doesn't mean that they're necessarily related in the way that we'd like.

In this class, we're going to study the **art** of proof. This is a subject that could easily take an entire textbook to develop; we limit ourselves to a few pages, in the interests of time and teaching by example.

1.1 The Art of Proof

In the above section, we came up with a reasonably rigorous definition of what makes up a proof:

1. A well-stated claim (i.e. one that contains all of the things we're assuming to make our claim true.)

2. A selection of statements we've previously proven true, along with perhaps some axioms.

3. A number of logical links between these statements, axioms, and assumptions that concludes that our claim must be true.

This definition indeed captures the letter of what it means to be a proof; however, it does not properly capture the **spirit** of what a proof should aspire to be! Consider the following example:

Claim 1.

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Proof.

$$\begin{aligned}
\sqrt{xy} &\leq \frac{x+y}{2} \\
xy &\leq \frac{(x+y)^2}{4} \\
4xy &\leq (x+y)^2 \\
4xy &\leq x^2 + 2xy + y^2 \\
0 &\leq x^2 - 2xy + y^2 \\
0 &\leq (x-y)^2.
\end{aligned}$$

□

This proof is **awful**. Why? Well, first and foremost, it has no words! In fact, we have absolutely no idea what we're even proving, nor any idea what x and y are supposed to be, nor any idea how the equations we've drawn are linked together. So: **never do this**! Whenever you're writing a proof, **use words**. Always tell your reader what you're proving, how you're going about making said proof, and how you're linking together any of these steps.

For example, the thing above is *supposed* to be a proof of the arithmetic-geometric mean inequality, which is the following claim:

Theorem 2. *(AM-GM) For any two nonnegative real numbers x, y, we have that the geometric mean of x and y is less than or equal to the arithmetic mean of x and y: in other words, we have that*

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

With this stated, we can then see the **second** flaw in the cautionary example above: strictly as written, it's not even a proof of the AM-GM! The failed proof above looks like it starts off by **assuming** that the AM-GM is true, and then deduces a statement that we already know to be true (any squared number is nonnegative.) This does not, **by any means**, prove the statement we are claiming!

For example, if we assume that 1=2, we can easily deduce a true statement by multiplying both sides by 0:

$$\begin{aligned}&1 = 2\\ \Rightarrow &0 \cdot 1 = 0 \cdot 2\\ \Rightarrow &0 = 0.\end{aligned}$$

Does this prove 1=2? No! As we stated above, proofs can only take in as admissible evidence **things we already know to be true**. In specific, to prove a statement is true, you can't, um, just assume that the statement is true.

In specific, what does this mean for our proof of the AM-GM? Well, it means that instead of starting with the AM-GM and deducing a true thing, we should start with some true things and then deduce that the AM-GM is a consequence of these true things. We present a fixed and fully functional proof here:

Theorem 3. *(AM-GM) For any two nonnegative real numbers x, y, we have that the geometric mean of x and y is less than or equal to the arithmetic mean of x and y: in other words, we have that*

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Proof. Take any pair of nonnegative real numbers x, y. We know that any squared number is nonnegative: so, in specific, we have that $(x-y)^2$ is nonnegative. If we take the equation $0 \leq (x-y)^2$ and perform some algebraic manipulations, we can deduce that

$$\begin{aligned}0 &\leq (x-y)^2\\ \Rightarrow 0 &\leq x^2 - 2xy + y^2\\ \Rightarrow 4xy &\leq x^2 + 2xy + y^2\\ \Rightarrow 4xy &\leq (x+y)^2\\ \Rightarrow xy &\leq \frac{(x+y)^2}{4}.\end{aligned}$$

Because x and y are both nonnegative, we can take square roots of both sides to get

$$\sqrt{xy} \leq \frac{|x+y|}{2}.$$

Again, because both x and y are nonnegative, we can also remove the absolute-value signs on the sum $x+y$, which gives us

$$\sqrt{xy} \leq \frac{x+y}{2},$$

which is what we wanted to prove. □

In terms of the formulas used, this proof is identical to the "awful" proof we had earlier; however, because we changed the ordering of these formulas and added a lot of discussion about precisely "what" we're trying to prove and why we can justify the steps we've made, this proof is a lot more satisfying and persuasive.

1.2 Pictures and Proofs

Words and symbols are not the only tool in proofs! In fact, well-chosen and drawn diagrams can often illustrate an idea that would otherwise take pages of text to describe. Pictures alone are rarely proofs: words are almost always necessary to explain what's going on, and you'll have to do some calculations to solve almost any problem. However, a well-placed picture can often be invaluable, as we demonstrate in the following example:

Claim 4. *For any $n \in \mathbb{N}$, we have the following identity:*

$$\sum_{k=1}^{n} \frac{1}{4^k} = \frac{1-(1/4)^n}{3}.$$

(The $\sum_{k=1}^{n}$-expression above is a shorthand way of writing the sum $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^n}$. In general, the $\sum$ symbol is used for this kind of shorthand, where we want to add up a bunch of objects but don't want to actually completely write out the sum each time.)

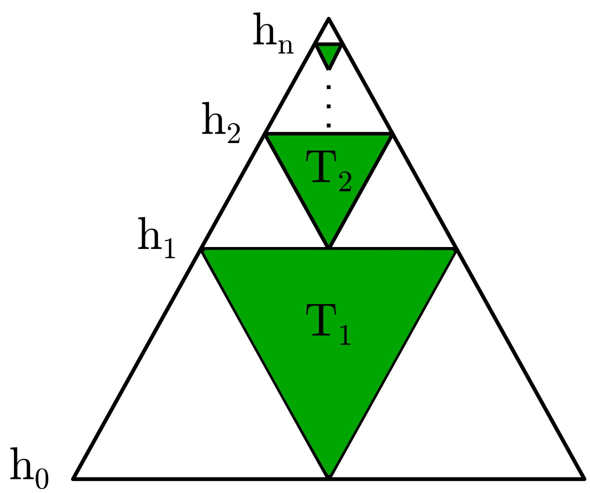
Proof. Consider the following construction:

1. Start by taking an equilateral triangle of area 1.

2. By picking out the midpoints of its three sides, inscribe withinin this triangle a smaller triangle T_1. Color this triangle green. Also, notice that by symmetry this green triangle has area $\frac{1}{4}$, as drawing it has broken up our original triangle into four identical equilateral triangles.

3. Take the "top" triangle of the three remaining white triangles, and repeat step 2 on this triangle. This creates a new green triangle, T_2, with area $\frac{1}{4}$ of the white triangle's area: i.e. $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

4. Keep repeating this process until we have drawn n green triangles, as depicted below:



5. What is the combined area of all of the green triangles? On one hand, we've seen that the area of each T_k is just $\left(\frac{1}{4}\right)^k$, as T_1 had area $\frac{1}{4}$ and each green triangle after the first had area $\frac{1}{4}$ of the green triangle that came before it. Summing over all of the green triangles, this tells us that

$$\text{Area(Green)} = \sum_{k=1}^{n} \frac{1}{4^k}.$$

6. On the other hand, as shown in our picture, we can see that between height h_0 and h_1, green triangles are taking up precisely a third of the area of our original area-1 triangle. Similarly, green triangles are taking up a third of the area from h_1 to h_2, h_2 to h_3, and so on/so forth all the way to h_n, after which there are no more green triangles.

   Therefore, the total area of the green triangles is just a third of the area of our original triangle that lies between height h_0 and h_n. Because the area of the last tiny white triangle at the top is (by construction) equal to the area of T_n, i.e. $\left(\frac{1}{4}\right)^n$, we then have that

$$\text{Area(Green)} = \frac{1}{3} \cdot \left(1 - \left(\frac{1}{4}\right)^n\right).$$

By combining these two expressions for the total area of the green triangles, we have proven that

$$\sum_{k=1}^{n} \frac{1}{4^k} = \frac{1-(1/4)^n}{3}.$$

□

1.3 Avoiding Overkill in Proofs

One last thing to mention in mathematics (that is particularly applicable to Mathcamp students) is the following bit of warning about "overkill" in proofs. Many of you have seen a lot of mathematics before: consequently, when you're going through this course, you're often going to be tempted to use tools you've seen in other math classes (most notoriously, L'Hôpital's rule) to attack problems. Be careful about doing this! While sometimes you can create some absolutely beautiful connections between your different classes by taking results from one and putting them in the other, at other times you may find yourself accidentally making problems trivial that would otherwise be fascinating by using a result that (is much more complex than the result you're studying / actually needs the proof of the problem you're studying in order to prove that result, so you'd be engaging in some circular logic).

For example, consider the following proof:

Theorem 5. *$\sqrt[3]{2}$ is irrational: i.e. there are no pair of positive integers p, q, $q \neq 0$, such that $\sqrt[3]{2}$ can be expressed as the fraction $\frac{p}{q}$.*

Proof. First, recall Fermat's Last Theorem, a result formulated in 1637 by the mathematician Pierre de Fermat and proven in 1995 by the mathematician Andrew Wiles, whose proof was the culmination of centuries of labor by scientists and mathematicians:

If n is a natural number ≥ 3, the equation

$$a^n + b^n = c^n$$

has no solutions with $a, b, c \in \mathbb{N}$.

We're going to use this to... prove that $\sqrt[3]{2}$ is irrational.

To do this, suppose that we have expressed $\sqrt[3]{2}$ as some ratio $\frac{p}{q}$, where p, q are a pair of positive real numbers. Then, if we cube both sides, we have

$$\frac{p^3}{q^3} = 2;$$

multiplying both sides by q^3 then gives us

$$p^3 = q^3 + q^3.$$

Fermat's last theorem says that such a thing cannot exist, if $p, q \in \mathbb{N}$; therefore, because Fermat's last theorem is true, we know that no matter how we've expressed $\sqrt[3]{2}$ as a ratio $\frac{p}{q}$, we can never have both p and q be positive integers. Therefore, $\sqrt[3]{2}$ must be an irrational number. □

This proof works completely! – and yet, by reading it, we really haven't gained any better insights into what makes a number irrational. Good proofs are ideally ones that **illuminate** the question at hand: not only do they rigorously show that the statement in question is true, they also shed light on how the concepts involved in the proof work, and how the reader might go about attacking similar problems.