Proof Techniques	Instructor: Padraic Bartlett
Homework 4	
Week 1	*Mathcamp 2012*

Homework instructions: many of the problems below are labeled with the tags $(*)$ or $(+)$. $(*)$ denotes that the problem in question is fairly fundamental to the topics we're studying and is something that you should make sure you understand completely, while $(+)$ denotes a problem that may be much harder than some of the others on the set.

This class is homework-required! What this means is that I'm expecting you to try every problem, to solve almost all of the $(*)$ ones, and most of the non-$(+)$ ones. The $(+)$ ones are certainly problems you are capable of solving, and I want you solve some of these! But they will not be as necessary for your ability to survive and thrive in later lectures, and I don't expect people to solve all of them. If you get stuck, or see a typo, find me! I can offer tons of hints and corrections. HW will be handed in at the start of class every week; I'll try to look over solutions in between classes, and come up with comments.

1. [$(*)$] Let x be an integer. Show that if x^2 is even, x must also be even.

2. [$(*)$] Find the flaw in the following proof:

   Theorem 1 *All ponies are the same color.*

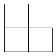
   Proof. We proceed by induction. Specifically, let $P(n)$ be the claim "In any collection of n ponies, all of these ponies are the same color."

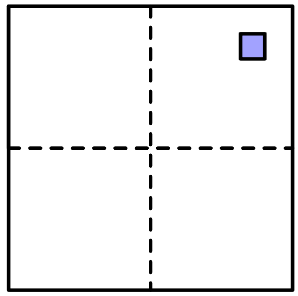
   Base case: we want to prove $P(1)$. But $P(1)$ is trivially true; in any collection made of one pony, all of the ponies in that set are the same color.

   Inductive case: we want to prove that $P(n) \Rightarrow P(n+1)$. In other words, we want to prove that whenever $P(n)$ is true, $P(n+1)$ is also true. To do this: assume that $P(n)$ is true, i.e. that in any set of n ponies, all of those ponies are the same color. With this assumption, we want to prove that $P(n+1)$ is true: i.e. that in any set of $n+1$ ponies, all of these ponies are also the same color.

   To do this: take any set of $n+1$ ponies, and write them as the set $\{p_1, \ldots p_{n+1}\}$. Break this set up into two subsets of size n: the subset $\{p_1, \ldots p_n\}$ and the subset $\{p_2, \ldots p_{n+1}\}$. These are both sets of size n: by our inductive hypothesis, they are both the same color. But these sets share the ponies $p_2, \ldots p_n$ in common! Therefore, whatever color our first set $\{p_1, \ldots p_n\}$ is **must be** the same color as the second set $\{p_2, \ldots p_{n+1}\}$, because they overlap! Therefore, all of our $n+1$ ponies are the same color, and we've proven that $P(n+1)$ is true (given our assumption $P(n)$.)

   So: we've proven that $P(1)$ is true, and that $P(n) \Rightarrow P(n+1)$. Therefore, by induction, we've proven that our claim $P(n)$ is true for all n; if we let n be the total number of ponies in existence, this proves our claim.

3. For any $n \in \mathbb{N}$, take a $2^n \times 2^n$ grid of unit squares, and remove one square from somewhere in your grid. Can you exactly cover all of the remaining squares using these  three-square tiles (along with their flips and rotations)?



4. [(+)] Some natural numbers can be expressed as a sum of smaller consecutive natural numbers: for example, we can write 31 as $15 + 16$, and 30 as $11 + 10 + 9$. Others cannot: for example, 32 cannot be written as the sum of smaller consecutive natural numbers! Which natural numbers n cannot be expressed in this way? Which can? (As always, prove your claim.)

5. [(∗)] Show that every number greater than 12 can be made from some combination of 4's and 5's.

6. Prove that every 4-th Fibonacci number is a multiple of 3. (Hint: show that $f_{4k+4} = 5f_{4k} + 3f_{4k-1}$, for any k.)

7. A **lattice path** in the plane $\mathbb{R}^2$ is a path joining integer points via steps of length 1 either upward or rightward. Show that for any $a, b \in \mathbb{N}$, the number of lattice paths from $(0, 0)$ to the point (a, b) is $\binom{a+b}{a}$.

8. [(∗)] Given five integer points[1] in the plane, prove that there is some pair of them such that the midpoint of the segment joining them is also an integer point.

9. [(∗)] Six people are sitting in a café. Prove that either three of them have never met each other, or three of them all know each other. (Assume that knowledge is symmetric: i.e. if A knows B, B also knows A.)

10. [(+)] In class, we proved that in any string of $n^2 + 1$ real numbers, there must be some monotone subsequence of length $n + 1$. Prove that this result is the best we could hope for, in the following sense: for any n, create a set of n^2 real numbers such that it does not contain any monotone subsequence of length $n + 1$.

11. Take five points within a square of side length 1. Prove that there must be two points that are within $\frac{\sqrt{2}}{2}$ of each other. Similarly, show that if you take nine points within a cube with side length 1, there must be two that are within $\sqrt{3}$ of each other.

[1] In other words, a point with coördinates (a, b), where a and b are both integers.