Proof Techniques — Instructor: Padraic Bartlett

Homework 3

Week 1 — *Mathcamp 2012*

Homework instructions: many of the problems below are labeled with the tags $(*)$ or $(+)$. $(*)$ denotes that the problem in question is fairly fundamental to the topics we're studying and is something that you should make sure you understand completely, while $(+)$ denotes a problem that may be much harder than some of the others on the set.

This class is homework-required! What this means is that I'm expecting you to try every problem, to solve almost all of the $(*)$ ones, and most of the non-$(+)$ ones. The $(+)$ ones are certainly problems you are capable of solving, and I want you solve some of these! But they will not be as necessary for your ability to survive and thrive in later lectures, and I don't expect people to solve all of them. If you get stuck, or see a typo, find me! I can offer tons of hints and corrections. HW will be handed in at the start of class every week; I'll try to look over solutions in between classes, and come up with comments.

1. [$(*)$] For any pair of real numbers a, b such that $a < b$, find a bijection between the sets $[0, 1]$ and $[a, b]$.

2. Find a bijection between the sets $(-1, 1)$ and $\mathbb{R}$.

3. [$(+)$] Find a bijection between the sets $[0, 1]$ and $(0, 1)$.

4. [$(*)$] Are the sets $\mathbb{N}$ and $\mathbb{N}^3$ (i.e. the set of all triples (a, b, c), where a, b, c are all natural numbers) the same size?

5. Let X denote the set made out of all possible sequences of natural numbers with finite length: i.e. for every element x of X, there is some length k such that x looks like some string $(n_0, n_1, \ldots n_k)$, where the $n_1 \ldots n_k$'s are all natural numbers. Is this set the same size as $\mathbb{N}$?

6. Let Y denote the set made out of all possible sequences of natural numbers with infinite length: i.e. for every element y of Y, y looks like some string $(n_0, n_1, \ldots)$, where the elements n_i are natural numbers. Is this set the same size as $\mathbb{N}$?

7. [$(+)$] Let A and B be a pair of sets, $f : A \to B$ an injective function, and $g : B \to A$ another injective function. Show that there is a bijection from A to B.

8. [$(+)$] A real number r is called **algebraic** if and only if there is some degree n and integer constants $a_0, \ldots a_n$ such that r is a root[1] of the following polynomial:

   $$a + 0 + a_1 x + \ldots + a_n x^n.$$

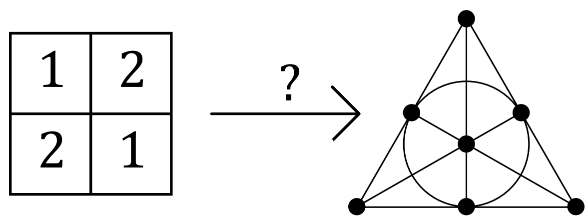
   Most numbers you know are algebraic: for example, all of $\mathbb{Q}$ is (they're roots of the polynomial $qx - p$), as is $\sqrt{2}$ (it's a root of $x^2 - 2$) and most other things.

   Show that $\mathbb{N}$ has the same cardinality (size) as $\mathcal{A}$, the collection of all algebraic numbers.

[1] A **root** of a polynomial is a number you can plug into that polynomial and get 0. For example, 2 is a root of the polynomial $x^2 - 4$, because plugging in 2 for x yields 0.

Latin squares!

On one hand: Latin squares can be used to do **anything**. You can use Latin squares to send encrypted messages, you can turn Latin squares to break large complicated graphs into lots of smaller ones, you can convert a Latin square into a schedule for running experiments with lots of potentially-correlated variables, and you can even turn sets of them into beautiful geometric pictures:



On the other hand: Latin squares are really just dumbed-down Sudoku grids (specifically, they're $n \times n$ arrays filled with the symbols $\{1, \ldots n\}$, such that no symbol is repeated twice in any row or column.)

So: how can such simple things have so many complicated and beautiful applications? In this class, we will study as many applications of Latin squares as we can; in theory, by the end you will both have a deep understanding of how Latin squares work alongside a broad idea of their applications across all of mathematics.