Latin Squares	Instructor: Padraic Bartlett
Lecture 4: Latin Squares and Geometry	
Week 3	*Mathcamp 2012*

In this week's lectures, we're going to shift our emphasis from the theory of Latin square to the applications of Latin squares: i.e. now that we understand these things a little, what can we actually **do** with them? Specifically, let's start by looking at how we can use Latin squares to do some **geometry**...

1 Affine Planes

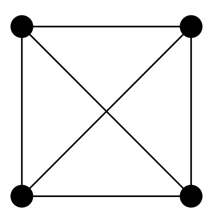
Definition. An **affine plane** is a collection of points and lines in space that follow the following fairly sensical rules:

(A1): Given any two points, there is a unique line joining any two points.

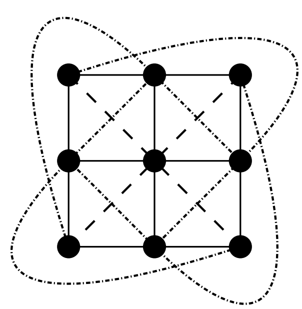
(A2): Given a point P and a line L not containing P, there is a unique line that contains P and does not intersect L.

(A3): There are four points, no three of which are collinear. (This rule is just to eliminate the silly case where all of your points are on the same line.)

$\mathbb{R}^2$ satisfies these properties, and as such is an affine plane. In this class, we're going to be interested in studying **finite** affine planes: i.e. affine planes with finitely many points. For example, the following set of four points and six lines defines an affine plane:



The following set of nine points and twelve lines defines another affine plane:



(All lines in the picture above contain three points. There are four curved tiny-dash lines, two long-dash diagonal lines, and six bold left-right and up-down lines.)

Finite affine planes satisfy a number of interesting properties. To better understand how these objects work, we prove a few of these properties here:

Proposition. In any affine plane, there is an integer n such that every line in our plane contains exactly n points, and every point lies on precisely $n+1$ lines. We call this value the **order** of our plane.

Proof. There are two possible cases to consider here:

1. Suppose that for any two lines L_1, L_2 in our plane, we can always find a third point P such that P does not lie on either of these lines. Then, given any point Q on the line L_1, we can find a line M through Q and P using property $A1$ of our affine plane. This line cannot intersect any other elements on L_1, because otherwise (if it did, at some point R) we would not have a unique line defined by the points Q and R (as both L_1 and M would contain both of them, while being distinct lines because M contains P while L_1 does not.)

   So, every point in L_1 is contained within exactly one line through P. Furthermore, there is exactly one other line that goes through P that intersects no point of L_1, by property $A2$. So, if $|L_1|$ denotes the total number of points contained in the line L_1, we have that $|L_1|+1$ many lines go through P.

   Similarly, every point in L_2 is contained within exactly one line through P, and there is precisely one other line through P that does not intersect L_2. Therefore, if $|L_2|$ denotes the total number of points contained in the line L_2, we also have that $|L_2|+1$ many lines go through P.

   But these two things are counting the same object: the number of lines through P. Therefore, these two quantities are equal: i.e. $|L_1| = |L_2|$. Therefore, all lines contain the same number (call it n) of points, and any point is contained by $n+1$ many lines.

2. If we are not in the first case, then there are two lines L_1, L_2 such that every point of P is contained within our two lines. We claim that our plane is in fact the affine plane with four elements that we gave an example of earlier.

   To see why: first, notice that by our property $A3$, we must have two of these points on L_1 and not on L_2, and the other two on L_2 and not on L_1. Call the L_1 points P_1, P_2 and the L_2 points Q_1, Q_2. Suppose for contradiction that we had a third point running around. By assumption, it has to lie on either line L_1 or L_2: without loss of generality, assume it lies on line L_1, and call it P_3.

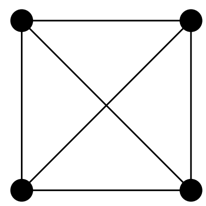
   Examine the line M_1 formed by the points P_1, Q_1, M_2 formed by the points P_2, Q_2 and M_3 formed by the points P_3, Q_2. Note that neither M_1 nor M_2 can be L_1 or L_2, by the argument we just made above.

   We know that at most one of the line M_2, M_3 can be parallel to M_1; therefore, at least one of them must intersect M_1.

   Suppose that M_1 and M_2 intersect at some point. If it is a point P_i on L_1, then the pair of points P_1, P_i defines both of the distinct lines M_1 and L_1, which contradicts our property $A1$. Instead, if it's a point Q_j on L_2, then the pair of points Q_1, Q_j also defines both of the distinct lines M_1 and L_2, which contradicts $A1$ again.

So we cannot have M_1 and M_2 intersecting; therefore, we must have M_1 and M_3 intersect. But this creates the same set of problems — no matter how they intersect, we'll get a pair of points that define two distinct lines!

Therefore, we must have that L_1 and L_2 must contain exactly two points, as must all other lines; consequently, we have that there are four points in total in our plane. Because any two of them uniquely define a line, we have $\binom{4}{2} = 6$ many edges in total, $\binom{3}{1} = 3$ of which pass through any point. This is in particular the affine plane we drew earlier with four points and six edges, which we call the affine plane of order 2.



Proposition. Any finite affine plane of order n contains n^2 many points.

Proof. By our earlier property, every point P is on $n+1$ many lines, each of which contains $n-1$ points distinct from P. By properties $A1$ and $A2$, there is exactly one line connecting any other point in the plane to P.

By combining these two results together, we can see that there are

$$(n+1)(n-1)+1 = n^2$$

many points in the plane, by simply using these $n+1$ lines to count all of the points other than P in the plane.

For our last property, the definition of a **parallel class** will be useful:

Definition. A **parallel class** in an affine plane is a collection of lines that are all parallel: i.e. such that no two of them intersect.

Proposition. Take any finite affine plane of order n. Then there are exactly $n^2 + n$ lines in this plane, which can be partitioned into $n+1$ distinct parallel classes, each of which contains n lines.

Proof. Pick any point P and any line L passing through P. Let M be any other line passing through P; then, for each of the $n-1$ non-P elements in M, there is a parallel line passing through that element parallel to L. By taking these n parallel lines along with the line L, this creates a parallel class with n many elements in it.

Do this for every line passing through P: this creates $n+1$ different parallel classes. Every line M shows up in exactly one parallel class, as (by $A2$) there is a unique line through P parallel to M, and that line determines which of the $n+1$ different parallel classes M is in. Therefore, this process counts each of our lines exactly once, and gives us n^2+n many lines in total.

Given the ease with which we were able to punch through the above properties, it seems like affine planes are things we should be able to understand; i.e. simple questions like when they exist and how to construct them easily should be well-understood. This, however, is completely false! As it turns out, we only know how to construct these objects for orders n where n is a prime power: i.e. a number of the form p^k, for some prime p and positive integer k.

This is surprising; at the same time, it should remind you of our results for sets of MOLS, for which we only knew how to construct sets of $n-1$ MOLS for prime power orders. Far more surprisingly, it turns out that these two questions are equivalent! In other words, a set of $n-1$ MOLS of order n is surprisingly the same thing as a finite affine plane of order n. We prove this here:

Theorem 1 *A finite affine plane of order n exists if and only if a set of $n-1$ MOLS of order n exist.*

Proof. We first describe how to turn a set of MOLS into an affine plane. To do this, use the following construction: for points, take all of the pairs (i, j), where $1 \leq i, j \leq n$. For lines, we list the lines of our affine plane in groups of n, corresponding to the $n+1$ parallel classes we showed must exist earlier:

- Given any i, all of the cells in row i form a line. The collection of these n lines is a parallel class.

- Given any j, all of the cells in row j form a line. The collection of these n lines forms another parallel class.

- Take any Latin square L_α of our $n-1$ MOLS. Given any symbol s, let all of the cells containing the symbol s in L_α be a line. The collection of all of these n lines, one for each symbol, forms a parallel class. We get $n-1$ such parallel classes, one for each Latin square in our set.

Because our squares are mutually orthogonal, none of these lines overlap. Therefore, given any point (i, j), we've actually just shown that it lies on $n+1$ lines, each of which contain $n-1$ other points: therefore, the collection of all of these lines collectively contains

$$(n+1)(n-1)+1 = n^2$$

many points. In other words, (i, j) is connected to every other cell in our Latin square by some line; therefore, we satisfy $A1$.

To see that we satisfy $A2$, take any line M and any other point (i, j) not on M. If M is a row, take the row i; this line is parallel to M, and is furthermore unique in doing so (as every column and every set of cells underlying a symbol in some L_α must eventually contain the row i.) Similarly, if M is a column, take the column j; this is also the unique line parallel to M through (i, j). Finally, if M is a set of symbols underlying some symbol s in the Latin square L_α, take the set of symbols underlying whatever symbol is in (i, j) in L_α.

This is parallel to M by construction, and is furthermore unique in doing so: given any other line N containing (i, j), N must either be a row or column (and therefore intersect

M) or must come from some other symbol t and other Latin square L_β, in which case it must intersect M (specifically, at whatever cell the overlap of L_α and L_β contains the pair (s,t). Such a cell must exist, because L_α and L_β are mutually orthogonal, and therefore every such pair comes up exactly once.)

Therefore, we satisfy $A2$. As long as we started with at least 1 MOLS of at least order 2, this has created at least 6 lines (at least 2 for rows, 2 for columns, and 2 for parallel classes) each with at least 2 points, we satisfy $A3$; so we satisfy $A3$, and are therefore an affine plane!

The reverse process is completely the same, but backwards. Specifically, given an affine plane of order n, split it up into $n+1$ parallel classes $C_0, \ldots C_n$ each with n elements. Number the elements in each class C_i 1 through n.

Unilaterally declare the parallel class C_0 to correspond to the rows of our Latin squares, and the parallel class C_n to correspond to the columns of our Latin square. To each of the coordinates (i,j), assign the unique point given by the intersection of the i-th line in the parallel class C_0 and the j-th line in the parallel class C_n. This creates a bijection between points in our affine space and cells (i,j).

Given any number k between 1 and $n-1$, we fill the Latin square L_k as follows: place the symbol s in the cell (i,j) if the line s of class C_k contains the point we identified with (i,j) earlier. Because every point is contained in some line of C_k (n disjoint lines each with n points), this fills every cell. As well, this preserves our Latin property, because any line from our class C_k shows up in any row (i.e. intersects any line from C_0) or any column (i.e. intersects any line from C_n) exactly once, by property $A2$.

This creates $n-1$ Latin squares. Furthermore, given any two such Latin squares L_α, L_β, and any two lines $s \in C_\alpha$, $t \in C_\beta$, we know that s and t intersect at exactly one point: i.e. there is exactly one cell in our square where L_α is s and L_β is t. In other words, every pair of symbols shows up exactly once: i.e. these squares are mutually orthogonal!

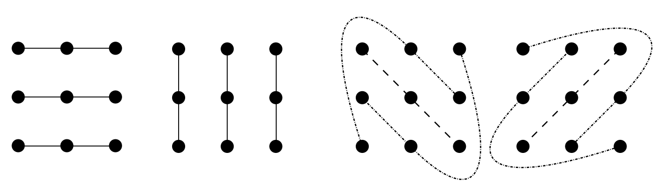
We draw this process here, for the case where n is 3:

Latin squares:

1	2	3
3	1	2
2	3	1

1	2	3
2	3	1
3	1	2

Corresponding parallel classes:



Affine plane: