The Unit Distance Graph | Instructor: Paddy

Lecture 4: "Nice" Colorings

Week 1 of 1 | *Mathcamp 2010*

1 Glossary

Surface A surface is just a collection of points that "locally" looks like $\mathbb{R}^2$; explicitly, a surface is a shape that you can get by gluing pairs of edges on a regular $2n$-polygon together.

Metric A metric, loosely speaking, is a function that defines the concept of distance on a space.

Embedding An embedding of a graph G on a surface S is a way of drawing G on S, so that all of the vertices of G are points on S and the edges of G are curves drawn on S.

Planar embedding A planar embedding is an embedding in which the curves for any two edges never intersect (except at possibly their endpoints.)

Nice n-coloring Take a surface with metric (S, d) and a graph G planarly embedded on S. A nice n-coloring is a way of painting the faces of G with n colors, so that no two faces within distance 1 of each other get the same color.

Simple closed curve A continuous map $\gamma : [0, 1] \to S$ such that $\gamma(0) = \gamma(1)$ and for any other pair of points $t, s \in (0, 1)$, $\gamma(t) \neq \gamma(s)$.

Contractible A simple closed curve γ is called contractible if it bounds a region in S that "looks like" an open disk. Rigorously: γ is contractible iff there is a continuous map $F : [0, 1]^2 \to S$ such that $F(s, 0) = F(s, 1) = \gamma(0)$, $F(0, t) = \gamma(t)$, $F(1, t)$ is the constant function $\gamma(1)$, and $F(s, t)$ never intersects γ whenever $s \neq 0$ and $t \neq 0, 1$.

Interior The interior of a contractible simple closed curve γ is the region that γ bounds that looks like an open disk: in other words, it's the region of S where $F(s, t)$'s values live.

Area For a subset A of our surface with metric (S, d), $area(A)$ is defined to be the maximal number of pairwise disjoint open discs of radius $1/2$ that we can completely fit in A. (This is not a completely standard definition, but it is useful here.)

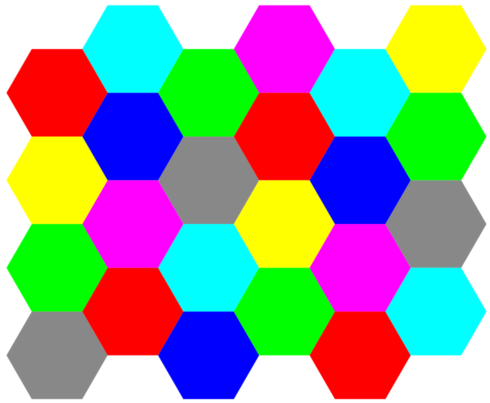
$D_n(x)$ For a graph G and vertex $x \in V(G)$, the set $D_n(x)$ consists of all of the elements $v \in G$ that are distance n from x: i.e. all vertices that have a walk of length n to x, but no walks of shorter length to x.

Locally finite A graph G is called locally finite if $D_1(x)$ is finite, for every $x \in G$.

Locally Hamiltonian A graph is called locally Hamiltonian if for every $x \in G$, there is a cycle in G made out of the vertices of $D_1(x)$.

2 Thomassen's 7CT

So: recall how we tiled the plane with hexagons to show that $\chi(\mathbb{R}^2) \leq 7$:



A natural question to ask, after seeing this coloring, is the following: can we do any better? In other words, suppose that we consider coloring the faces of some planar graph G on $\mathbb{R}^2$, and we concern ourselves with not just avoiding monochromatic edges of length 1, but making sure that no two faces that lie within distance 1 of each other receive the same color. Can we come up with a 6-coloring?

The answer (perhaps surprisingly) is no! In fact, suppose that we don't concern ourselves with just the plane, but in fact with any surface S with a metric d. Then, we have the following property:

Theorem 1 *Suppose that S is a surface and k is a natural number with the following properties:*

1. *Every noncontractible simple closed curve has diameter ≥ 2.*
2. *Every simple closed curve C with diameter < 2 is such that the area of $int(C)$ is $\leq k$.*
3. *The diameter of S is $\geq 12k + 30$.*

Take any graph G that can be planarly embedded on S. Then we need at least 7 colors to nicely color the faces of G.

Proof. We first open with a remarkably useful lemma:

Lemma 2 *If G is a connected, locally finite, locally Hamiltonian graph with at least 13 vertices, then G has a vertex of degree at least 6.*

Proof. Suppose not: that all of the vertices of G have degree ≤ 5. Then, we have one of the following five cases (follow along with pen and paper!):

- $\Delta(G) = 1$. In this case, G consists of pairs of edges and isolated vertices, and is clearly not connected; a contradiction.

- $\Delta(G) = 2$. In this case, because G is connected, it must be C_{13}; consequently, G is not locally Hamiltonian (as the neighbors of any two vertices aren't connected.)

- $\Delta(G) = 3$. Pick x such that $\deg(x) = 3$, and look at $D_1(x)$, x's neighbors. Because G is locally Hamiltonian, these are all connected in a cycle; hence, we have that $x \cup D_1(x)$ is a tetrahedron, and thus disconnected from the rest of G.

- $\Delta(G) = 4$. Pick x such that $\deg(x) = 4$, and again look at $D_1(x)$. Because G is still locally Hamiltonian, we have that $D_1(x)$ is connected in a cycle, and thus that each vertex in $D_1(x)$ can be connected to at most one vertex in $D_2(x)$. Because we have more than 5 vertices in G, at least one vertex $y \in D_1(x)$ is connected to some vertex $z \in D_2(x)$. Then, because $D_1(y)$ *also* has to form a cycle, we have that there are edges from z to both of y's neighbors in $D_1(x)$; consequently, no other vertices than z can live in $D_2(x)$. But this means that for $D_1(z)$ to contain a cycle, we need to have z connected to all four elements in $D_1(x)$; so all of the vertices of G have degree 4, and thus no more vertices can be added! But this graph has only 6 vertices; a contradiction.

- $\Delta(G) = 5$. Again, pick x such that $\deg(x) = 5$. Because of the edges used in our locally Hamiltonian condition, each element of $D_1(x)$ has no more than two neighbors in $D_2(x)$. As well, each element in $D_2(x)$ has at least two neighbors in $D_1(x)$ and two more in $D_2(x)$, again because of the locally Hamiltonian condition; so each element in $D_2(x)$ can be connected to at most 1 element in $D_3(x)$. Suppose that there is some element in $D_3(x)$, and call it z; then, again by the locally-Hamiltonian property, z has at least three neighbors in $D_2(x)$.

  So: because there are at most 5 vertices in $D_1(x)$, there are likewise at most 5 vertices in $D_2(x)$ by degree considerations; thus, before we start looking at edges to $D_3(x)$, we know that there are already 4 edges tied up for every vertex in $D_2(x)$. So, there are at most 5 edges spare from $D_2(x)$ to $D_3(x)$; as every vertex uses at least 3 edges, this means that $D_3(x)$ has at most 1 vertex. Using our locally-Hamiltonian property for the last time, we have at last that $D_4(x)$ must be empty, and thus that G can have at most $|D_0(x)| + |D_1(x)| + |D_2(x)| + |D_3(x)| = 1 + 5 + 5 + 1 = 12$ vertices, a contradiction.

(It bears noting that the above logic also shows that the icosahedron is the unique connected locally finite, locally Hamiltonian graph that's 5-regular.)

So: why did we prove this lemma?

Well: let's proceed by contradiction. Assume we have a pair (G, S) that satisfies our conditions and admits a nice 6-coloring. Consider the dual graph $M(G, S)$ (sometimes called the map graph) to our graph G, formed by taking G's faces as our vertices and connecting two such vertices iff their faces intersect in G.

Take any vertex x in M, and let C_x be the cycle of edges in G that bounds the face marked by x. Fix some orientation of C_x, and let $x_1, \ldots x_n$ be the list of faces in G that we visit when going around C_x.

Suppose for a moment that all of these vertices were distinct, for every x. Then we would have that the neighbors of x, $x_1 \dots x_n$, form a cycle in M for every x – i.e. that M is locally Hamiltonian! Thus, by our lemma, we would have that M must contain a vertex v of degree at least 6.

Examine the face corresponding to v in G. It must have diameter < 1, if we're in a nice coloring; as well, it needs to have at least six neighboring faces. Consequently, because we've given G a nice coloring, v's face and all of its neighboring faces must be different colors – i.e. we need at least 7 colors to nicely color G!

The rub, then, lies in the fact that it's entirely possible for a vertex to be repeated in C_x, as we show below:

So: what do we do? Let's start by considering some such "bad" case: i.e. let x, y be vertices such that y shows up in two nonconsecutive entries of C_x. Let e_i and e_j be two such nonadjacent edges, and let R be a simple closed curve in the faces bounded by C_x and C_y, that crosses each of e_i, e_j precisely once and crosses no other edges. Because the diameter of any face is < 1, the diameter of R is < 2 and thus R is contractible.

Consequently, by the Jordan curve theorem, R divides M into two pieces: the interior of R and the exterior of R. As a result, we have that that $M - \{x, y\}$ is disconnected!

How many vertices live inside of R? Well: for every i-colored vertex z in M that lies in $int(R)$, pick some point $s_z \in S$ from inside of the face labeled by z. Because the area contained by R is $\leq k$, there are at most k such vertices; finally, because we're assuming that we have a nice 6-coloring, we have at most 6 such collections of colored vertices. Consequently, $int(R)$ contains at most $6k$ vertices.

So: look at the connected components of the graph $M - \{x, y\}$. Each one that corresponds to the interior of some R has $\leq 6k$ vertices, as we've just shown: how about the rest? Well, first notice the following pair of observations:

- There is a component of $M - \{x, y\}$ that is *not* the interior of some R. This is because the diameter of any such R is ≤ 2, and each such R starts somewhere in the face given by x; consequently, because the diameter of our space is $\geq 12k + 30$, we must have some faces not contained within a R.

- This component is in fact unique! To see this, let $e_1, \dots e_n$ list the edges in order around C_x, and $f_1, \dots f_m$ list the edges in $C_x \cap C_y$ in the order that they appear in when going around C_x. Each pair of these edges f_2, f_{1+1} define a curve R like the one we've considered above. Thus, for each such curve $R_{i,i+1}$ through f_i and f_{i+1}, we have that R contains all of the faces between some e'_i and some e'_j in either the clockwise or counter-clockwise orientation, depending on how R goes around x.

So: what would happen if there were two nonadjacent edges e_i, e_j in C_x such that a loop R never goes around them? Well, because y is a single continuous face, we know that there couldn't be a xy-loop on at least one side of the e_i, e_j's: so the faces corresponding to e_i and e_j are connected. So there's exactly 1 connected component that's not in the interior of some R.

So: call such a pair $\{x, y\}$ a 2-separator, and denote the collection of all of the interior components of loops R constructed above by writing $int(M, x, y)$. Take the collection of all such pairs $\{x, y\}$ where $M - \{x, y\}$ is disconnected. Discard any pairs $\{x, y\}$ where either x or y lie strictly within $int(M, u, v)$ for some other pair $\{u, v\}$. Call this collection of pairs we now have a collection of *maximal 2-separators*, and call the edge xy of any maximal 2-separator a *crucial edge*.

Consider the subgraph H formed in G by deleting $int(M, x, y)$ for every maximal 2-separator $\{x, y\}$. H is connected, as the shortest path between any two elements in H will never go through $int(M, x, y)$ (as it's always faster to go through the edge xy than anything in the interior.) As well, because M has diameter $> 12k + 30$, we know that there must be at least 7 vertices in H (as the diameter of any $int(M, x, y)$ is always bounded by 4, as it's constrained by the diameters of the loops R in x, y.)

So: we claim that this graph H is locally Hamiltonian! To see this: take any vertex $x \in H$, and look at its neighbors $x_1, x_2, \ldots x_k$. If these are all distinct, we're done! So suppose not; that $x_i = x_j$, for $i \neq j, j \pm 1$. Then the pair $\{x, x_j\}$ is a 2-separator; because it's still in our graph, it's a maximal 2-separator, and thus we removed the interior $int(M, x, x_j)$ from our graph. Thus, we know in fact that all of the vertices between x_i and x_j are the same and equal to x_j! So in fact we have that this is a cycle.

So: by the lemma, because this is a connected locally finite, locally Hamiltonian graph, either it has a vertex of degree 6 (in which case we're done, as before) or every vertex is of degree ≤ 5 and there are no more than 12 vertices. Consequently, we have no more than 30 edges in our graph H.

So: how do we turn H back into M? Well: all we have to do is simply "glue" the interiors of $int(M, x, y)$ back in along the crucial edges $\{x, y\}$.

We claim that doing this increases the diameter of H to $< 12k + 30$. Why? In the worst-case scenario, we attached two $int(M, x, y)'s$ to vertices that are on completely opposite sides of H, which has diameter < 30. Each of $int(M, x, y)$'s components have $< 6k$ vertices; consequently, in the worst-case scenario we have a path in our graph from a vertex in one component to a vertex in another component of length $12k + 30$, which forces our surface to have strictly smaller diameter (as the diameter of any face is < 1.)

Thus, we have that S has diameter $< 12k + 29$, a contradiction. So a vertex of degree 6 must exist! Consequently, no nice 6-coloring can exist, as claimed.