Lecture 1: The Unit Distance Graph Problem; König's Lemma

1 The Unit Distance Graph Problem

Question 1 *Consider the following method for turning $\mathbb{R}^2$ into a graph:*

- *Vertices: all points in $\mathbb{R}^2$.*
- *Edges: connect any two points (a, b) and (c, d) iff the distance between them is exactly 1.*

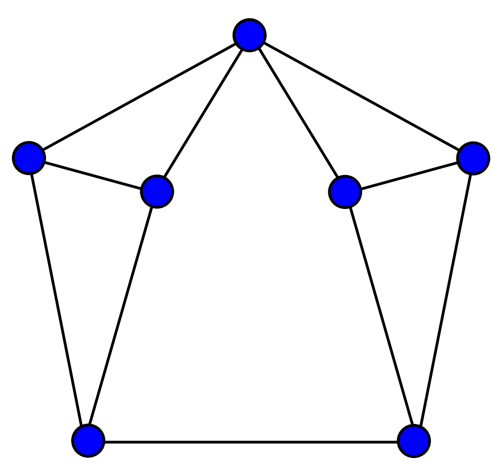
What is the chromatic number of this graph?

How can we even bound such a thing? Well: to get a lower bound, it certainly suffices to consider finite graphs G that we can draw in the plane using only straight edges of length 1 – as our graph on $\mathbb{R}^2$ must contain any such graph as a subgraph, examining these graphs will give us some easy lower bounds!

So, by examining a equilateral triangle T, which has $\chi(T) = 3$, we can see that

$$\chi(\mathbb{R}^2) \geq 3;$$

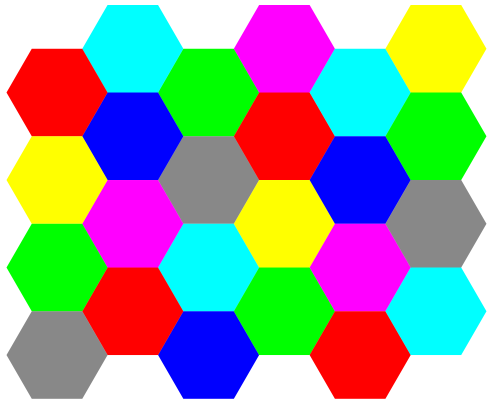
similarly, by examining the following pentagonal construction (called a Moser spindle,)



we can actually do one better and say that

$$\chi(\mathbb{R}^2) \geq 4.$$

Conversely: to exhibit an upper bound on $\chi(\mathbb{R}^2)$ of k, it suffices to create a way of "painting" the plane with k-colors in such a way that no two points distance 1 apart get the same color. Thus, by considering the following way of tiling the plane with hexagons,



we can see that seven colors are sufficient: i.e. that

$$\chi(\mathbb{R}^2) \leq 7.$$

These bounds on $\chi(\mathbb{R}^2)$ took us a little more than a page to accomplish; as a result, we might hope that completely resolving this question is something we could easily finish within the lecture! (After all, it's a relatively simple question: how hard can it be to find the chromatic number of the plane, anyways?)

As it turns out: really, really hard. This problem – often called the Hadwiger-Nelson problem in graph theory literature – has withstood attacks from the best minds in combinatorics since the 1950's. In this class, we will detail several approaches people have taken towards resolving this problem. Specifically, we hope to discuss the following topics:

- König's lemma – how to apply it to the unit distance graph question.
- The chromatic number of $\mathbb{Q}^2$ (and in general, $\mathbb{Q}^n$).
- The possible dependence of $\chi(\mathbb{R}^2)$ on the axiom of choice!
- "Nice" colorings of the plane, and Thomassen's 7-color-theorem

Chili levels will be around 2-2.5 for the first three lectures, and 3-3.5 for the last two (as there's a lot of elegant topological notation that we'll need to discuss first.)

2 Konig's Lemma

Lemma 2 *Suppose that (T, t_0) is a rooted tree at t_0 on $\aleph_0$-many vertices, and suppose that the degree of every vertex is finite. Then there is an infinite descending path in T starting at t_0.*

Proof. For ease of notation: let T_v denote the tree acquired by taking v and all of the paths that descend from v in our tree T.

We create our path by induction. Start at t_0.

If we've made it to some vertex v: let $v_1, \ldots v_n$ be the descendants of v, and take as our inductive hypothesis the claim that T_v has infinitely many vertices in it. Then the trees

$T_{v_1}, \dots T_{v_n}$ form a partition of the vertex set of $T_v \setminus \{v\}$: as this is an infinte set of vertices, the pigeonhole principle tells us that one of these trees must contain infinitely many vertices! Let T_{v_i} be that vertex, and go to v_i.

Repeating this process yields an infinte path descending through T, starting at t_0.

This might seem like a rather trivial proof: the depth of its consequences, then, might surprise you:

Corollary 3 *Suppose that G is a graph on $\aleph_0$-many vertices, such that any finite subgraph of G can be k-colored. Then G can be k-colored.*

Proof. Create a tree as follows: Enumerate the vertices of G as $\{v_i\}_{i=1}^{\infty}$, and let the levels L_n of our tree be given by the collection of all k-colorings of $\{v_1 \ \dots v_n\}$. Draw an edge from any k-coloring of $\{v_1 \ \dots v_n\}$ to a coloring of $\{v_1 \ \dots v_{n+1}\}$ iff the coloring of $n+1$-vertices extends our coloring of n vertices.

This is then a tree! It has infinitely many vertices, and the degree of any vertex is finite; thus, by König's lemma, there's an infinite path! This path – made of colorings that all agree with each other – then gives us a k-coloring of all of G.

The above observation motivates the following question: for countable graphs, to demonstrate k-colorability, it suffices to simply work on the collection of all finite graphs. Does the same hold for uncountable graphs? Specifically, to find the chromatic number of the plane, does it suffice to create an upper bound on all finite graphs embedded in the plane, with edges given by straight line segments of length 1?

It turns out that the answer here is yes! In HW #2, in fact, you will prove that this result stems from Zorn's lemma: in the meantime, consider that this remarkable result gives us good reason to consider the following definition:

Definition. A graph G has Euclidean dimension n iff for every $m \geq n$, G can be embedded in $\mathbb{R}^m$ so that any two points $x, y \in \mathbb{R}^m$ are connected by an edge iff they're distance 1 apart.

Using the language of this definition, the theorem that you'll show on HW #2 can be phrased as the following:

Corollary 4 *$\chi(\mathbb{R}^2)$ is equal to the maximum chromatic number of the collection of graphs with Euclidean dimension 2.*

Consequently, to find the chromatic number of the plane, it suffices to just understand the chromatic number of graphs with Euclidean dimension 2! Many current paths that people are taking to resolving the unit distance graph problem go through the path of classifying graphs of Euclidean dimension 2; if you're curious about understanding the Euclidean dimension of graphs further, I've got lots of handouts from one of the currently-running projects for you.