Lecture 2: Evans' Conjecture

Our goal in the next two lectures will be to prove the following result, which (until its proof) was known as Evans' Conjecture:

Theorem 1 *(Smetianuk, 1981) Any $n \times n$ partial latin square with $\leq n-1$ entries can be completed to a latin square.*

This theorem breaks into two cases, each of which takes about a lecture; specifically, we can consider separately the cases where there are $\leq n/2$-distinct symbols amongst our $n-1$ completed entries, or the case where there are more than $n/2$-distinct symbols amongst our entries. Today, we will prove the first case:

Theorem 2 *Any $n \times n$ partial latin square P with $\leq n-1$ entries and $\leq n/2$ distinct symbols in P can be completed to a latin square.*

Proof. As described in our earlier lecture, for any such partial latin square P we can exchange the rows and symbols of P to get a partial latin square with $\leq n-1$ entries in which at most $n/2$ rows are nonempty. By then permuting the rows of P, we can insure that all of our entries lie in the first r rows of P, for $r \leq n/2$. Furthermore, if $f_i =$ the number of completed elements in row i, we can use more row-permutations to force

$$f_1 \geq f_2 \geq \ldots f_r.$$

We now claim that we can complete this to a $r \times n$ latin rectangle, expanding this one row at a time. How can we do this?

Well: just like before, we're motivated to try to use Hall's marriage theorem, which says

Theorem 3 *Suppose that $G = (A, B)$ is a bipartite graph that satisfies* ***Hall's property****:*

$$(\ddagger): \qquad \forall H \subset A \text{ or } H \subset B, |N(H)| \geq |H|.$$

Then G has a 1-factor.

Specifically, we'd like to use the following alternate form of Hall's marriage theorem, which (HW) you're encouraged to prove if you haven't yet seen it:

Theorem 4 *Suppose that $G = (A, B)$ is a bipartite graph that satisfies*

$$(\star): \qquad \forall H \subset A, |N(H)| \geq |H|, \text{ and } |A| = |B|.$$

Then G has a 1-factor.

So: suppose that we've filled in all of the rows $1, \ldots l-1$ in P. Consider the following bipartite graph (A, B):

- A_i = the collection of all elements not used thus far in column i,
- X = the collection of all elements $\{1, \ldots n\}$ that aren't already used in row l.
- A = the collection of all of the A_i, where i is a column that doesn't contain an already filled-in entry of the l-th row.

If we can find a 1-factor between the A_i's and X, we can use this to fill in all of the remaining entries in the l-th row, in a way that preserves our latin-square properties! Thus, we merely need to do this first for $l = 1$, and then repeatedly increment l; this will eventually complete our partial latin square P to a $r \times n$ latin rectangle, which we can then complete by our results yesterday! Thus, it suffices to find such a 1-factor.

We first note the following useful lemma:

Lemma 5

$$n - f_l - l + 1 > l - 1 + (f_{l+1} + \ldots + f_r).$$

Proof. For $l = 1$, this is just

$$\begin{aligned} & n - f_1 > +(f_2 + \ldots + f_r) \\ \Leftrightarrow & n > (f_1 + f_2 + \ldots + f_r), \end{aligned}$$

which we know to be true by assumption.

For $l > 1$: we have

$$\begin{aligned} & n - f_1 > +(f_2 + \ldots + f_r) \\ \Rightarrow & n > \sum_{i=1}^{n} f_i \geq (l-1)f_{l-1} + f_l + \ldots f_r, \end{aligned}$$

because $f_1 + \ldots f_{l-1} \geq f_{l-1} + \ldots f_{l-1} = (l-1)f_{l-1}$.

There are then two possibilities: either

- $f_{l-1} \geq 2$, in which case we have

$$\begin{aligned} & n > (l-1)f_{l-1} + f_l + \ldots f_r \\ \Rightarrow & n > (l-1)2 + f_l + \ldots f_r \\ \Rightarrow & n - f_l - l + 1 > f_{l+1} + \ldots f_r. \end{aligned}$$

- $f_{l-1} = 1$. In this case, we have that f_k is forced to be ≤ 1, for every $k \geq l - 1$; consequently, we're just trying to show that

$$\begin{aligned} & n - 1 - l + 1 > l - 1 + (r - l) \\ \Leftrightarrow & n > r + l - 1. \end{aligned}$$

  Because $l \leq r \leq n/2$, this holds as well! So our lemma is proven.

Returning to our proof: by Hall's marriage theorem it suffices to show that for any collection $A_{j_i}, \dots A_{j_m}$, we have

$$\left| N\left(\bigcup_{i=1}^{m} A_{j_i}\right)\right| = \left|\bigcup_{i=1}^{m} A_{j_i}\right| \geq m.$$

For ease of notation, denote $\bigcup_{i=1}^{m} A_{j_i}$ by B.

How can we do this? Well: let c denote the number of cells in the columns $j_1 \dots j_m$ that lie in X. Then, there are $\leq (l-1)\cdot m$ cells in X used in all of the rows above the l-th row (as we're assuming that they're all full, and there are m columns and $l-1$ rows.) As well, there are $\leq f_{l+1} + \dots f_r$-many cells in X used in the rows below l, as those are the only filled cells below the l-th row: so we have that

$$c \leq (l-1)m + f_{l+1} + \dots + f_r.$$

Conversely: pick any $x \in X$ that doesn't lie in B; then, by definition, it must show up in every single column $j_i, i = 1 \dots m$, somewhere. Because there are m different columns, we then have that each such x shows up in c m different times: thus, we have that

$$m(|X| - |B|) \leq c.$$

Combining, we have that

$$\begin{aligned} m(|X| - |B|) &\leq (l-1)m + f_{l+1} + \dots + f_r \\ \Rightarrow |B| &\geq \frac{mn - mf_l - (l-1)m - (f_{l+1} + \dots f_r)}{m} \end{aligned}$$

We're trying to show that $|B| > m-1$; so, it'd definitely suffice to show that

$$\begin{aligned} \frac{mn - mf_l - (l-1)m - (f_{l+1} + \dots f_r)}{m} &> m-1 \\ \Rightarrow m(n - f_l - l + 2 - m) &> f_{l+1} + \dots + f_r. \end{aligned}$$

So: consider various cases. If $m = 1$ or $m = n - f_l - l + 1$, this is just our lemma. Consequently, we have that this holds for every value in between, as the LHS of the above equation is a quadratic polynomial in m with m^2's coefficient $= -1$.

It then suffices to consider $m > n - f_l - l + 1$.

So: pick any $x \in X$. We know that x is in $\leq l-1+f_{l+1}+\dots f_r$ rows in P: ergo, it is also in $\leq l-1+f_{l+1}+\dots f_r$ many columns. So, if we have $\geq n-f_l-l+1 > l-1+f_{l+1}+\dots f_r$-many A_j's, there's always an A_j that contains x! So $|B| = |X| \geq m$.

Thus, our graph satisfies Hall's property; so we have a 1-factor, and thus we can complete this row! Repeating this process yields a latin rectangle, which we can then complete to a latin square.