Lecture 5: Ramsey Theory

Question 1 *Amongst any collection of 6 people, can you always find three mutual friends or three mutual strangers?*

Solution. In the language of graph theory, our question is the following: if you (possibly improperly) color the edges of K_6 red and blue, do you always have to create a monochrome triangle?

We claim that you will always do so. To see why: pick any red-blue coloring of K_6, and any vertex $v \in K_6$. There must be three edges leaving v of one of our two colors, as $deg(v) = 5$; suppose without loss of generality that those three edges are red. Let $\{w_1, w_2, w_3\}$ be the endpoints of these edges.

Then, there are two cases:

- There is some edge $\{w_i, w_j\}$ that's red. In this case, the vertices v, w_i, w_j form a red triangle.
- Every edge $\{w_i, w_j\}$ is blue. In this case, the vertices w_1, w_2, w_3 form a blue triangle.

A natural generalization of the above question is the following:

Question 2 *Let $C = \{1, \dots c\}$ be some collection of colors, and $n_1, \dots n_c$ integers. Is there always some value of n such that if K_n is colored with c colors, then K_n necessarily contains a i-monochrome K_{n_i}, for some $1 \leq i \leq c$?*

Theorem 3 *(Ramsey's Theorem) The answer to this question is yes!*

Proof. Let $R(n_1, \dots, n_c)$ denote the smallest value of n such that if K_n is colored with c colors, then K_n necessarily contains a i-monochrome K_{n_i}. We seek to show that R is well-defined.

To do this, we proceed by induction on the number of colors $C = \{1, \dots, c\}$. When $c = 1$, note that this is trivial, as $R(k) = k$ for all k.

Now, consider the two-color case. We trivially have $R(n, 1) = R(1, n) = 1$ and $R(n, 2) = R(2, n) = n$. Furthermore, we claim that we have the following recursive bound on the growth of $R(r, s)$:

$$R(r, s) \leq R(r, s-1) + R(r-1, s).$$

To see why: take a complete graph K on $(R(r, s-1) + R(r-1, s))$ many vertices, and color its edges red and blue (or 1 and 2, if you prefer integers). Pick any $v \in K$, and partition the rest of K's vertices into two sets:

- B', which contains all of the vertices in K connected to v by a blue edge, and

- R', which contains all of the vertices in K connected to v by a red edge.

Let B and R be the subgraphs of K induced by these vertices, respectively.

Because K has

$$R(r, s-1) + R(r-1, s) = |B| + |R| + 1$$

many vertices, either $|B| \geq R(r, s-1)$ or $|R| \geq R(r-1, s)$. If the former, then induction on the values of r and s tells us that we either have

- a blue K_s inside of B, or
- a red K_{r-1} inside of R, in which case we have that there's a red K_r inside of $v \cup R$;

consequently, we're done! (Analogous reasoning applies to the case $|R| \geq R(r-1, s)$.)

So: now suppose that we've settled our theorem for $c-1$ colors. We seek to resolve it for c colors. Specifically, we make the following claim:

$$R(n_1, \ldots n_c) \leq R(n_1, \ldots n_{c-2}, (R(n_{c-1}, R_c))).$$

To see this: let K be the complete graph on $R(n_1, \ldots n_{c-2}, (R(n_{c-1}, R_c)))$ many vertices, and color K's vertices with c different colors.

Now: become selectively colorblind! In other words, pretend temporarily that $c-1$ and c have the same colors.

Then, by our inductive hypothesis, either

- there is an i-monochrome K_{n_i}, for $1 \leq i \leq c-2$, or
- there is a $(c\&c-1)$-colored $K_{R(n_{c-1}, n_c)}$. By the definition of $R(n_{c-1}, n_c)$, this means that there's either a $(c-1)$-monochrome $K_{n_{c-1}}$, or a c-monochrome K_{n_c}.

So we're done!

In the language of the proof above, the opening question for this lecture can be thought of as showing $R(3,3) = 6$.

To illustrate a few ideas that go into finding a Ramsey number, and to maybe illustrate some of the difficulty of finding such numbers, consider the following question:

Question 4 *What's $K_{3,4}$?*

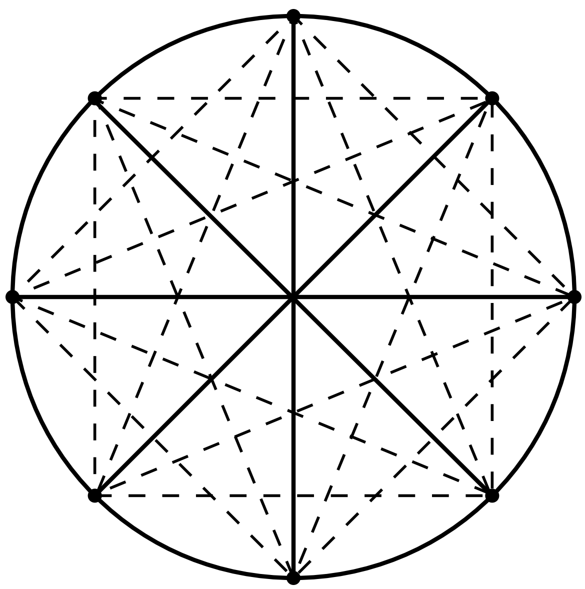
Solution. Pick n such that for any red-blue coloring of K_n, we have neither a blue K_3 nor a red K_4. Pick any $x \in K_n$, and again let

- B be the subgraph induced by the set of vertices in K_n connected to v by a blue edge, and
- R be the subgraph induced by the set in K_n connected to v by a red edge.

If there is a blue edge in B, then $x \cup B$ will yield a blue K_3; similarly, if there is a red K_3 in R, $x \cup R$ yields a red K_4. Because $R(2,4) = 4$ and $R(3,3) = 6$, we have that if neither situation occurs, we must have $|B| \leq 3$ and $|R| \leq 5$. In other words, we've just shown that for any vertex $x \in K_n$, we have $deg_b(x) \leq 3$ and $deg_r(x) \leq 5$. Consequently, the total degree of x must be ≤ 8; i.e. $n \leq 9$, and thus $R(3,4) \leq 10$.

Consider the case $n = 9$. In this case, each x must have $deg_b(x) = 3$ and $deg_r(x) = 5$; consequently, the number of blue edges in K_n can be counted, via the degree-sum formula, to be $\frac{1}{2} \sum_{x \in K_n} deg_b(x) = 27/2 = 13.5$. Since we can't have half of a blue edge, this is also impossible! So $R(3,4) \leq 9$.

Conversely: consider the following drawing below. The solid edges form a graph with girth ≥ 4, and so do not contain a K_3. As well, picking any four points on the boundary of a 8-cycle necessarily involves picking two opposite points or two adjacent points; so there is no complete K_4 amongst 4 points within the dashed edges.



Thus, $R(3,4) > 8$; i.e. $R(3,4) = 9$.