Graph Colorings	Instructors: Marisa and Paddy

Lecture 3: Marriage and the Edge Chromatic Number

Week 1 of 1	*Mathcamp 2010*

1 Glossary

Matching A matching H of a graph G is a 1-regular subgraph of G.

Line Graph The line graph $L(G)$ of a graph G is the graph with vertex set given by the edges of G, and an edge $\{e, f\}$ in G iff these two edges are incident in G.

Edge Coloring A n-edge coloring of a graph G is a mapping from the set $E(G)$ into the set $\{1, 2, \ldots n\}$ such that no two incident edges receive the same colors.

Edge Chromatic Number The edge chromatic number of a graph G, $\chi'(G)$, is the smallest value of n such that G admits a n-edge coloring.

2 Hall's Marriage Theorem

Theorem 1 *Take a bipartite graph $G = (A, B)$. Then, the following conditions are equivalent:*

- *G has a 1-factor.*
- *(Hall's condition): For any subset $X \subset A$ or $X \subset B$, if $N(X)$ denotes the neighbors of X, then $|X| \leq |N(X)|$.*

Proof. ($\Rightarrow$): Suppose that G has a 1-factor; because G is bipartite, such a 1-factor is just a pairing-up of vertices in A and in B along edges in G. Thus, for any subset $X \subset A$, because $N(X)$ must contain the edges in this 1-factor, we have that $|X| \leq N(X)$ (and similarly for $X \subset B$.)

($\Leftarrow$): Take any matching M in G. Consider the following algorithm for creating an alternating path between distinct vertices in A and B:

1. Suppose without loss of generality that M's not already a 1-factor, and pick some $a_0 \in A$ that's nor involved in M.
2. Suppose that the sequence $a_0 b_1 a_1 b_2 a_2 \ldots b_{k-1} a_{k-1}$ has been created. Then, because the set $\{a_0 \ldots a_{k-1}\}$ of chosen A-vertices is strictly larger than the set $\{b_1 \ldots b_{k-1}\}$of chosen B-vertices, there must be some element $b \in B$ that's connected by some edge $\{b_i, a\}$ to some previously-chosen a_i, by Hall's condition. Let b_k be equal to b, and define $f(k) = i$ (so that $\{b_k, a_{f(k)}\}$ is the edge we used here to pick b.).

3. If b_k is in M, let a_k be the vertex across from b_k in M, and return to (2) to continue to grow our sequence. Otherwise, end our sequence! By construction, we know that b_k is unique amongst the previously chosen b_i's; similarly, because we picked the a_i up to this point by using the matching M, we know that they're all distinct. So this is still a sequence of distinct vertices!

4. Let the sequence that this algorithm terminates with be denoted as $a_0 b_1 \ldots b_{k-1} a_{k-1} b_k$. Notice, now, that this sequence of vertices, by construction, have the following properties:

   - a_0 and b_k are both unmatched.
   - b_i is adjacent to some element in $\{a_0 \ldots a_{i-1}\}$.
   - $a_i b_i$ is in M, for all i.

5. So: consider the following path, made by alternately following the edges of M and the edges recorded by the function f:

$$bb_k, \{b_k, a_{f(k)}\}, a_{f(k)}, \{a_{f(k)}, b_{f(k)}\}, b_{f(k)}, \{b_{f(k)}, a_{f^2(k)}\} a_{f^2(k)}, \ldots \{b_{f^n(k)}, a_{f^{n+1}(k)}\},$$

   where $a_{f^{n+1}(k)} = a_0$.

   All of the edges $\{a_{f(k)}, b_{f(k)}\}$ lie in M, while none of the edges $\{b_k, a_{f(k)}\}$ lie in M, by construction. So, replace the collection of $\{a_{f(k)}, b_{f(k)}\}$ in M with the collection of $\{b_k, a_{f(k)}\}$ edges! This collection has precisely one more edge than the old collection, and only deals with the vertices a_0, b_k (which weren't in M anyways) and $a_1 \ldots a_{k-1}, b_1 \ldots b_{k-1}$ (which were involved in edges we removed from M – so it preserves M's status as a matching! Thus, repeating this process will allow us to grow M into a 1-factor.

Corollary 2 *A k-regular bipartite graph $G = (A, B)$ can be decomposed into k disjoint 1-factors.*

Proof. Pick any subset $X \subset A$ or $X \subset B$ of size n. Because G is k-regular, there are kn distinct edges leaving X and entering $N(X)$. Consequently, as each vertex has degree k, there must be at least n vertices in $N(X)$ to absorb these edges! – so $|N(X)| \geq |X|$.

Thus, by Hall's Marriage theorem, there is a 1-factor in G. Deleting it from G leaves a $k-1$-regular graph; so repeating this process leaves us with a decomposition of G into k distinct 1-factors.

3 Edge Colorings

Proposition 3 *A cycle C_n has edge-chromatic number $\chi'(G) = \chi(G)$.*

Proof. Take a cycle C_n, and consider its line graph $L(C_n)$. This is another cycle! In fact, it's the same cycle as G, as it has the same number of vertices; thus, its edge chromatic number is the same as G.

Theorem 4 *If $G = (A, B)$ is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Proof. As we showed earlier in lecture, a k-regular bipartite graph G can be decomposed into k disjoint 1-factors. Simply coloring each of these 1-factors a different color, then, will insure that we have a k-edge-coloring of G, as no 1-factor contains two incident edges (by definition.)

So, it suffices to show that we can embed any bipartite graph G with maximum degree $\Delta(G)$ as a subgraph of some $\Delta(G)$-regular bipartite graph (as a k-edge coloring of a graph gives, by restriction, a k-edge coloring of all of its subgraphs.) To do this,

- simply add vertices to either A or B so that both sides have the same number of vertices, and then
- take any vertex $a \in A$ that doesn't have degree $\Delta(G)$. Then, because the number of edges leaving A is the same as the number of edges entering B, and all vertices have degree $\leq \Delta(G)$, there must be some vertex b in B also with degree $< \Delta(G)$. Add an edge between these two vertices! Repeat this process until the graph is $\Delta(G)$-regular.