Lecture 3: Nonstandard Dice

Definition. Define a k-die as a k-sided shape on which symbols $s_1, \dots s_k \in \mathbb{N}^+$ are drawn. Analogously, we can define a k-die to be a bucket with k balls in it, each stamped with a symbol $s_i \in \mathbb{N}^+$. In this sense, "rolling" our die corresponds to picking a ball out of our bucket; for intuitive purposes, pick whichever model makes more sense and feel free to use it throughout this lecture.

Question 1 *Given a k-die D, let a_n denote the number of ways in which rolling D yields a n. What is the generating function of $\{a_n\}_{n=0}^{\infty}$?*

Answer: Just

$$\sum_{n=0}^{\infty} a_n x^n,$$

where a_n can be thought of as the number of balls in D's bucket with a n stamped on them.

Definition. A standard k-die is a k-die for which $s_i = i$, for every i.

Question 2 *If D is a standard k-die, what is D's generating function?*

Answer: D has exactly one ball labeled s_i for every $1 \leq i \leq k$, and no other balls. Consequently, we can write D's generating function as

$$S_k(x) = \sum_{n=1}^{k} x^n = x \cdot \sum_{n=0}^{k-1} x^n = x \cdot \frac{x^k - 1}{x - 1}.$$

Question 3 *Let D_1 and D_2 be a pair of dice, and let a_n denote the number of ways in which rolling both dice can yield a pair of numbers that sum to n. What is the generating function for $\{a_n\}_{n=0}^{\infty}$?*

Answer: Let D_1's generating function be given by

$$A_1(x) = \sum_{n=0}^{\infty} b_n x^n,$$

and D_2's generating function be given by

$$A_2(x) = \sum_{n=0}^{\infty} c_n x^n.$$

If we roll both D_1 and D_2 and get a sum of n, we know that D_1 has to have came up k and D_2 has to have came up $n-k$, for some k. For a fixed value of k, there are precisely $b_k c_{n-k}$ ways in which this can happen; thus, the generating function for rolling both dice is given by

$$A_{1,2}(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} b_k c_{n-k} \right) x^n.$$

But this is just the product of the two power series $A_1(x)$ and $A_2(x)$! So, we have that

$$A_{1,2}(x) = A_1(x) \cdot A_2(x);$$

in other words, to get the generating function for the sum of two dice, we can simply take the product of their individual generating functions! (It bears noting that this will trivially generalize to rolling n dice at once.)

So, we've finally cleared enough ground to state the driving question of this talk:

Question 4 *For given k, n in $\mathbb{N}$, are there n nonstandard k-dice $D_1, \ldots D_n$ with generating functions $A_1(x), \ldots A_n(x)$ such that*

$$D_1(x) \cdot \ldots \cdot D_n(x) = \left(x \cdot \frac{x^k - 1}{x - 1} \right)^n ?$$

Equivalently: for given k and n, are there n nonstandard k-dice that have the same sum probabilities as a collection of n standard k-dice?

Answer: First, notice that the equivalence of the above statements stems from the observation that if two generating functions are equal, then all of their coefficients are equal; consequently, having the same generating functions is equivalent to having the same sum probabilities.

What are we trying to do? Well, we want to factor the polynomial $\left(x \cdot \frac{x^k-1}{x-1} \right)^n$ into n distinct polynomials $A_1(x), \ldots A_n(x)$ that are each equivalent to a nonstandard k-die. So, what conditions do we then need on the $A_i(x)$ to insure that they are nonstandard k-dice?

1. x is a factor of $A_i(x)$ – i.e. $A_i(0) = 0$. This insures that it is a die, as (by definition) we don't allow symbols that correspond to 0 in our dice.

2. $A_i(x)$ corresponds to a die with k faces: i.e. $A_i(1)$, the sum of all of the coefficients of $A_i(x)$, is k.

3. $A_i(x) \neq x \cdot \frac{x^k-1}{x-1}$. This insures that it's nonstandard.

By condition 1 above, we know that each $A_i(x)$ takes one of the x's in $\left(x \cdot \frac{x^k-1}{x-1} \right)^n$. Thus, our question really boils down to finding ways to factor $\frac{x^k-1}{x-1}$ into distinct parts!

How can we do this?

Deus Ex Machina 5

$$\frac{x^k - 1}{x - 1} = \prod_{d|k, d>1} \Phi_d(x),$$

where the polynomials $\Phi_d(x)$ *are the cyclotomic polynomials*

$$\Phi_d(x) = \prod_{\omega} (x - \omega),$$

where the product above is taken over all primitive d*-th roots of unity*[1]

Deus Ex Machina 6 *The cyclotomic polynomials are all irreducible.*

Deus Ex Machina 7 *The following is a list of some of the cyclotomic polynomials:*

$$\begin{aligned}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2 x + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(X) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1 \\
\Phi_9(x) &= x^6 + x^3 + 1 \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1
\end{aligned}$$

The proofs of the above observations are really beautiful and far beyond the reach of this course; but their results are fantastically useful! For example, we have the immediate observation:

Corollary 8 *If* k *is prime, no collection of* n *nonstandard* k*-dice can have the same sum probability as a collection of* n *standard* k*-dice.*

Proof. Because k is prime, it has no nontrivial factors beyond itself; so, by the above,

$$\frac{x^k - 1}{x - 1} = \prod_{d|k, d>1} \Phi_d(x) = \Phi_k(x)$$

is irreducible. Thus, by our earlier observations, there's no way to factor $\left(x \cdot \frac{x^k - 1}{x - 1}\right)^n$ into anything that's not a set of n standard k-dice!

[1] A primitive d-th root of unity is a complex number ω such that $\omega^d = 1$ and $\omega^i \neq 1$, for any $1 \leq i < d$.

Proposition 9 *If $k = 6$ $n = 2$, then there is precisely one pair of nonstandard 6-dice with the same probability as a pair of standard 6-dice.*

Proof. So: by the above, we seek to factor

$$\begin{aligned}\left(x\frac{x^6-1}{x-1}\right)^2 &= (x\Phi_2(x)\Phi_3(x)\Phi_6(x))^2\\ &= x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2\end{aligned}$$

into two polynomials $A_1(x), A_2(x)$ such that

1. $A_i(0) = 0$,
2. $A_i(1) = 6$, and
3. $A_i(x) \neq x(x+1)(x^2+x+1)(x^2-x+1)$.

So: because $x+1$ is 2 at $x = 1$, x^2+x+1 is 3 at $x = 1$, and x^2-x+1 is 1 at $x = 1$, we know that each $A_i(x)$ has to have exactly one copy of both $x+1$ and x^2+x+1 in it in order for $A_i(1)$ to be 6.

Consequently, the only way we can have both of these dice not be standard is if

$$\begin{aligned}A_1(x) &= x(x+1)(x^2+x+1)(x^2-x+1)^2 = x^8+x^6+x^5+x^4+x^3+x,\\ A_2(x) &= x(x+1)(x^2+x+1) = x^4+2x^3+2x^2+x;\end{aligned}$$

i.e. we have one die with faces $\{8,6,5,4,3,1\}$ and one die with faces $\{4,3,3,2,2,1\}$.