# Lecture 2: Binomial Coefficients and Stirling Numbers



**Question 1** *Let $f(n,k)$ denote the number of ways of picking $k$ elements out of the set $\{1,2,\dots n\}$. What is the generating function for $f(n,k)$ if we fix $n$? How about $k$? What if we fix neither? What's an explicit form for $f(n,k)$?*

**Answer**: So: temporarily, for the purposes of this question, forget that we know that $f(n,k) = \binom{n}{k}$. How can we create a generating function for these objects?

Well: first, notice that we have the recurrence relation

$$f(n,k) = f(n-1,k) + f(n-1,k-1).$$

Why is this? Well, pick some way of choosing $k$ elements out of $\{1,2,\dots n\}$. There are two possibilities: either we picked $n$, or we didn't! If we did, then ignoring the $n$ gives us a way of picking $k-1$ elements out of a set of $n-1$ objects; if we didn't, then we simply picked $k$ objects out of a set of $n-1$ elements. Summing over all of the ways of choosing $k$ elements out of $\{1,2,\dots n\}$ then gives us our desired result.

Also: notice that $f(n,0) = 1$, for all positive $n$, (as there's always exactly one way to not pick anything from a set), that $f(n,k) = 0$ for all negative $n,k$ (as there's no way to pick a negative number of things, or have a set with a negative number of elements,) and that $f(n,k) = 0$ if $k > n$ (as there's no way to pick more than $n$ things out of a set of $n$ elements.)

So: look at the generating function acquired by fixing $n$,

$$B_n(x) = \sum_{k=0}^{\infty} f(n,k)x^k.$$

Applying our recurrence relation to the above, then, yields

$$\begin{aligned}
B_n(x) &= \sum_{k=0}^{\infty} f(n,k)x^k \\
&= \sum_{k=0}^{\infty} (f(n-1,k) + f(n-1,k-1))x^k \\
&= \sum_{k=0}^{\infty} f(n-1,k)x^k + x\sum_{k=0}^{\infty} f(n-1,k-1)x^{k-1} \\
&= \sum_{k=0}^{\infty} f(n-1,k)x^k + \sum_{k=-1}^{\infty} f(n-1,k)x^k \qquad \text{(b/c } f(-1,k)=0) \\
&= B_{n-1}(x) + xB_{n-1}(x) \\
&= (1+x)B_{n-1}(x).
\end{aligned}$$

Then, because $B_0(x) = 1$ (shown via our boundary conditions,) we have via induction that

$$B_n(x) = (1+x)^n.$$

Using this, then, we can find $f(n,k)$ by extracting the coefficient of $x^k$ in this power series! To do this,

- simply take $k$ derivatives of $B_n(x)$ to kill off all of the terms with degree $< k$,
- evaluate the resulting power series at 0 to eliminate all of the terms with degree $> k$, and finally
- divide by $k!$ to cancel out the constant factor acquired by taking $k$ derivatives of $x^k$.

(It bears noting that this process will work on any power series! As such, it's a useful trick to have up your sleeve.)

So: doing this to $B_n(x)$ yields the following:

$$\begin{aligned}
\frac{d^k}{dx^k}\left(B_n(x)\right)\Big|_0 \cdot \frac{1}{k!} &= \frac{d^k}{dx^k}\left((1+x)^n\right)\Big|_0 \cdot \frac{1}{k!} \\
&= (n)(n-1)\cdots(n-k+1)\cdot(1+x)^{n-k}\Big|_0 \cdot \frac{1}{k!} \\
&= (n)(n-1)\cdots(n-k+1)\cdot(1)\cdot\frac{1}{k!} \\
&= \frac{(n)(n-1)\cdots(n-k+1)}{k!} \\
&= \frac{n!}{k!\cdot(n-k)!}
\end{aligned}$$

So: we've rederived the binomial coefficient! Awesome.

However: when we were deciding on a generating function to use, we chose to arbitrarily fix $n$ and look at $f(n,k)$ as a function of $k$. Why not $n$? Or, for that matter, why not both?

Well: one answer is that it seemed easier to deal with a fixed $n$, because it made our sum finite and fairly simple. However, it bears noting that we can easily do this in either way! For example, consider the **multivariable generating function** for $f(n,k)$ given by

$$C(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n,k)x^k y^n.$$

What is this function?

Well: by grouping terms, we have

$$\begin{aligned} C(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n,k)x^k y^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} f(n,k)x^k\right) y^n \\ &= \sum_{n=0}^{\infty} (1+x)^n\, y^n \\ &= \frac{1}{1-y(1+x)} \end{aligned}$$

(In general, we define a multivariable ordinary generating function for some sequence $f(n,k)$ precisely as we just did above.)

So: we can find an elegant form for the multivariable generating function for $f(n,k)$. How about the generating function

$$A_k(y) = \sum_{n=0}^{\infty} f(n,k)y^n?$$

Well: re-examine $C(x,y)$. By rearranging sums, we have that

$$\begin{aligned} C(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n,k)x^k y^n \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f(n,k)y^n x^k \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} f(n,k)y^n\right) x^k \\ &= \sum_{k=0}^{\infty} A_k(y)x^k. \end{aligned}$$

So: the generating function $A_k(y)$ is just the coefficient of $x^k$ in $C(x,y)$ – something we can easily find! Specifically, let the expression $[x^k]g(x)$ denote the coefficient of $x^k$ in the formal power series denoted by $g(x)$. Then, we have that

$$\begin{aligned} A_k(y) &= [x^k]\sum_{k=0}^{\infty} A_k(y)x^k \\ &= [x^k]\frac{1}{1-y(1+x)} \\ &= [x^k]\frac{1}{1-y}\frac{1}{1-(\frac{y}{1-y}(x)} \\ &= \frac{1}{1-y}[x^k]\frac{1}{1-(\frac{y}{1-y}(x)} \\ &= \frac{1}{1-y}[x^k]\sum_{n=0}^{\infty}\left(\frac{y}{1-y}\right)^n x^n \\ &= \frac{1}{1-y}\left(\frac{y}{1-y}\right)^n \end{aligned}$$

So:

**Question 2** *Similarly: let* $\left\{ n \atop k \right\}$ *denote the number of ways of partitioning the set* $\{1, 2, \ldots n\}$ *into* $k$ *nonempty pieces. What is the generating function for* $\left\{ n \atop k \right\}$ *if we fix* $n$*? How about* $k$*? What if we fix neither?*

**Answer**: So, the methods we use here are almost identical to the ones we developed above.

First, we develop a recurrence relation for these numbers. Take any partition $P$ of $\{1, \ldots n\}$ into $k$ pieces. Then, one of two cases occur:

- $\{n\} \in P$. In this case, deleting $n$ from our partition leaves a partition of $\{1, \ldots n-1\}$ into $k-1$ parts.
- In $P$, $n$ is in a set with other elements. In this case, deleting $n$ leaves a partition of $\{1, \ldots n-1\}$ into $k$ pieces; furthermore, we acquire each such partition in $k$ different ways, as $n$ can be live in any of the $k$ pieces of $P$, and deleting $n$ in any of these $k$ situations will always result in the same partition of $\{1, \ldots n-1\}$.

Thus, we have the recurrence relation

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k\left\{ n-1 \atop k \right\}.$$

Again, as before, we have (by considering various edge cases) $\left\{ 0 \atop 0 \right\} = 1$, $\left\{ n \atop 0 \right\} = 0$, and $\left\{ n \atop k \right\} = 0$ whenever $n < 0, k < 0$.

We now have 3 possible generating functions to consider:

$$A_n(y) = \sum_{k=0}^{\infty} \left\{ {n \atop k} \right\} y^k$$

$$B_k(x) = \sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} x^n$$

$$C(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left\{ {n \atop k} \right\} x^n y^k.$$

Which should we choose? Well: one strong motivation for studying $B_k(x)$ is that it will allow us to fix $k$, which should make applying our recursion much easier! We apply the recursion below:

$$\begin{aligned} B_k(x) &= \sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} x^n \\ &= \sum_{n=0}^{\infty} \left( \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \right) x^n \\ &= \sum_{n=0}^{\infty} \left\{ {n-1 \atop k-1} \right\} x^n + k \sum_{n=0}^{\infty} \left\{ {n-1 \atop k} \right\} x^n \\ &= x B_{k-1}(x) + kx B_k(x) \\ \Rightarrow \qquad B_k(x) &= \frac{x}{1-kx} B_{k-1}(x) \end{aligned}$$

Again, applying $B_0(x) = 1$, we have finally that

$$B_k(x) = \frac{x^k}{(1-x) \cdot (1-2x) \cdots (1-kx)}$$

So: we have a generating function! On the HW this week, we explore how to use this to find an exact form for $\left\{ {n \atop k} \right\}$, and what happens if we try to look at some of the other generating functions for the Stirling numbers of the second kind – so attempt those problems if you're still curious!