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Lecture 8: Integration Techniques, L'Hopital, and the Integral Test

Week 8 — *Caltech 2012*

1 A Curious Substitution: The Development

So: often, when we're integrating things, we often come up across expressions like

$$\int_0^{\pi} \frac{1}{1+\sin(\theta)} d\theta, \text{ or } \int_{-\pi/4}^{\pi/4} \frac{1}{\cos(\theta)} d\theta,$$

where there's no immediately obvious way to set up the integral. Sometimes, we can be particularly clever, and notice some algebraic trick. For example, to integrate $\sec(\theta)$, we can do the following:

$$\begin{aligned}\frac{1}{\cos(\theta)} &= \frac{\cos(\theta)}{\cos^2(\theta)}\\ &= \frac{\cos(\theta)}{1-\sin^2(\theta)}\\ &= \frac{1}{2}\left(\frac{\cos(\theta)}{1-\sin(\theta)} + \frac{\cos(\theta)}{1+\sin(\theta)}\right),\end{aligned}$$

which is now an integral we can study with u-substitutions.

Relying on being clever all the time, however, is not usually a winning strategy. It would be nice if we could come up with some way to methodically study integrals like the one above – specifically, to work with integrals that feature a lot of trigonometric identities! Is there a way to do this?

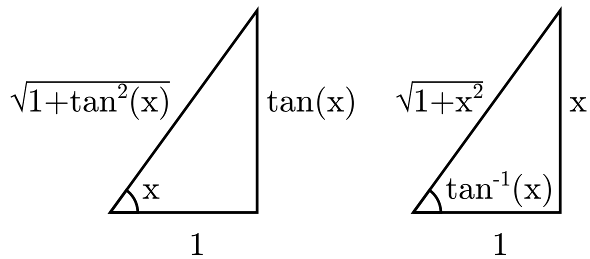
As it turns out: yes! Specifically, consider the use of the following function as a substitution:

$$g(x) = 2\arctan(x),$$

where $\arctan(x)$ is the inverse function to $\tan(x)$, and is a function $\mathbb{R} \to (\pi/2, \pi/2)$. In class, we showed that such inverse functions of differentiable functions are differentiable themselves: consequently, we can use the chain rule and the definition of the inverse to see that

$$\begin{aligned}&(\tan(\arctan(x))' = (x)' = 1, \text{ and}\\ &(\tan(\arctan(x))' = \tan'(\arctan(x)) \cdot (\arctan(x))' = \frac{1}{\cos^2(\arctan(x))} \cdot (\arctan(x))'\\ \Rightarrow &\frac{1}{\cos^2(\arctan(x))} \cdot (\arctan(x))' = 1\\ \Rightarrow &(\arctan(x))' = \cos^2(\arctan(x)).\end{aligned}$$

Then, if we remember how the trigonometric functions were defined, we can see that (via the below triangles)



we have that

$$(\arctan(x))' = \cos^2(\arctan(x)) = \frac{1}{1+x^2}$$

and thus that

$$g'(x) = \frac{2}{1+x^2}.$$

As well: by using the above triangles, notice that

$$\begin{aligned}\sin(g(x)) &= \sin(2\arctan(x)) \\ &= 2\sin(\arctan(x))\cos(\arctan(x) \\ &= 2 \cdot \frac{1}{\sqrt{1+x^2}} \cdot \frac{x}{\sqrt{1+x^2}} \\ &= \frac{2x}{1+x^2},\end{aligned}$$

and

$$\begin{aligned}\cos(g(x)) &= \cos(2\arctan(x)) \\ &= 2\cos^2(\arctan(x)) - 1 \\ &= \frac{2}{1+x^2} - 1 \\ &= \frac{1-x^2}{1+x^2}.\end{aligned}$$

Finally, note that trivially we have that

$$g^{-1}(x) = \tan(x/2),$$

by definition.

What does this all mean? Well: suppose we have some function $f(x)$ where all of its terms are trig functions – i.e. $f(x) = \frac{1}{1+\sin(x)}$, or $f(x) = \frac{1}{\cos(x)}$ – and we make the substiution

$$\int_a^b f(x) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(x))g'(x).$$

What do we know about the integral on the right? Well: as we've just shown above, the substitution of $g(x)$ turns all of the $\sin(x)$'s into $\sin(g(x))$'s, which are just reciprocals of polynomials; similarly, we've turned all of the $\cos(x)$'s into $\cos(g(x))$'s, which are also made of polynomials. In other words, this substitution turns a function that's made entirely out of trig functions into one that's made **only out of polynomials!** – i.e. it turns trig functions into quadratic polynomials! This is excellent for us, because (as you may have noticed) it's often *far* easier to integrate polynomials than trig functions.

This substitution is probably one of those things that's perhaps clearer in its use than its explanation. Consequently, we have several examples in the next section to illustrate how this substitution is used:

2 A Curious Substitution: Some Examples of its Use

Examples. Find the integral

$$\int_0^{\pi/2} \frac{1}{1+\sin(x)} dx.$$

Proof. So: without thinking, let's just try our substitution, where $f(x) = \frac{1}{1+\sin(x)}$:

$$\begin{aligned}
\int_0^{\pi/2} \frac{1}{1+\sin(x)} dx &= \int_{g^{-1}(0)}^{g^{-1}(\pi/2)} f(g(x))g'(x)dx \\
&= \int_{\tan(0)}^{\tan(pi/4)} \frac{1}{1+\frac{2x}{1+x^2}} \cdot \frac{2}{1+x^2} dx \\
&= \int_0^1 \frac{2}{1+x^2+2x} dx \\
&= \int_0^1 \frac{2}{(1+x)^2} dx \\
&= \int_1^2 \frac{2}{x^2} dx \\
&= -\frac{2}{x}\bigg|_1^2 \\
&= 1/2.
\end{aligned}$$

...so it works! Without any effort, we were able to just mechanically calculate an integral that otherwise looked nigh-impossible. Neat! □

However, it bears noting that this substitution is not a miracle worker; there are many functions whose integrals it will not simplify, and indeed some functions which it will make much more complicated. For these reasons, consider it a substitution of “last resort” – if you can’t think of anything else to try, go for it, but be aware that it can make some simple integrals far more complex than they need be (as we will see in our last example:)

Examples. Find the integral

$$\int_0^{\pi/2} \sin^2(x)dx.$$

Proof. Suppose that we have forgotten all about the double-angle formula, and just wanted to blindly apply our formula: then, for $f(x) = \sin^2(x)$, we would have

$$\begin{aligned}
\int_0^{\pi/2} \sin^2(x)dx &= \int_{g^{-1}(0)}^{g^{-1}(\pi/4)} f(g(x))g'(x)dx \\
&= \int_0^1 \left(\frac{2x}{1+x^2}\right) \cdot \frac{2}{1+x^2} dx \\
&= \int_0^1 \frac{8x^2}{(1+x^2)^3} dx,
\end{aligned}$$

which is arguably a much more awful thing to study! As it turns out, we can integrate it via partial fractions:

$$\begin{aligned}
\int_0^1 \frac{8x^2}{(1+x^2)^3} dx &= \int_0^1 \left(\frac{8}{(1+x^2)^2} - \frac{8}{(1+x^2)^3}\right) dx \\
&= \int_0^1 \left(\frac{8}{(1+x^2)^2} - \frac{8}{(1+x^2)^3}\right) dx,
\end{aligned}$$

which we can calculate via the u-substitution $x = \tan(u), dx = \frac{1}{\cos^2(x)}$:

$$\begin{aligned}
&\int_0^1 \left(\frac{8}{(1+x^2)^2} - \frac{8}{(1+x^2)^3} \right) dx \\
=&\int_0^{\pi/4} \left(\frac{8}{(1+\tan^2(u))^2} \cdot \frac{1}{\cos^2(u)} - \frac{8}{(1+\tan^2(u))^3} \cdot \frac{1}{\cos^2(u)} \right) du \\
=&\int_0^{\pi/4} \left(\frac{8}{\cos^{-4}(u)} \cdot \frac{1}{\cos^2(u)} - \frac{8}{\cos^{-6}(u)} \cdot \frac{1}{\cos^2(u)} \right) du \\
=&\int_0^{\pi/4} \left(8\cos^2(u) - 8\cos^4(u) \right) du \\
=&\int_0^{\pi/4} \left(4 + 4\cos(2u) - 2(1+\cos(2u))^2 \right) du \\
=&\int_0^{\pi/4} \left(4 + 4\cos(2u)) - 2 - 4\cos(2u) - 2\cos^2(2u) \right) du \\
=&\int_0^{\pi/4} (2 - 1 - \cos(4u))\, du \\
=&\left(u - \frac{\sin(4u)}{4} \right) \Bigg|_0^{\pi/4} \\
=&\pi/4.
\end{aligned}$$

Just to check, this does agree completely with the far easier method of just using the double-angle formula:

$$\int_0^{\pi/2} \sin^2(x) dx = \int_0^{\pi/2} \frac{1-\cos(2x)}{2} dx = \frac{x}{2} - \frac{\cos(2x)}{4} \Bigg|_0^{\pi/2} = \pi/4.$$

So: yeah, sometimes this method is a really bad idea. But sometimes (as in the two earlier examples) it's awesome! So don't be afraid to use it, but keep in mind that it's not a panacea; there are many many things that it will not help with. But some things that it will! □

3 Logarithmic Differentiation

As you may have noticed in this course, the logarithm and exponential functions show up everywhere in calculus; they have a remarkable variety of applications to problems, one of which is the process of **logarithmic differentiation**, which it is perhaps easiest to illustrate with an example:

Examples. Calculate

$$(x^x)'.$$

Proof. We first note, as a warning, that techniques like the power rule ($(x^n)' = nx^{n-1}$) are completely useless here, as it only works for x raised to some constant power – not a variable!

What can we do? Well: what happens if we take the log of x^x? We get $x \cdot \ln(x)$, which is something we do know how do take a derivative of. Can we use this to take a derivative of x^x itself?

As it turns out, yes! Specifically, note that by applying the chain rule on the left and the product rule on the right, we have

$$\begin{aligned} & (\ln(x^x))' = (x \cdot \ln(x))' \\ \Rightarrow\ & \frac{1}{x^x} \cdot (x^x)' = x \cdot \frac{1}{x} + 1 \cdot \ln(x) \\ \Rightarrow\ & \frac{1}{x^x} \cdot (x^x)' = 1 + \ln(x) \\ \Rightarrow\ & \qquad (x^x)' = x^x \cdot (1 + \ln(x)) \end{aligned}$$

□

As it turns out, we can pretty much always use the trick above to take derivatives. Specifically, suppose that we have some function of the form

$$f(x)^{g(x)}.$$

Then, if we take the log of this function, we have

$$\ln\left(f(x)^{g(x)}\right) = g(x) \cdot \ln(f(x));$$

if we then take derivatives of both sides using the chain and product rules, we finally have

$$\begin{aligned} & \left(\ln\left(f(x)^{g(x)}\right)\right)' = (g(x) \cdot \ln(f(x)))' \\ \Rightarrow\ & \frac{1}{f(x)^{g(x)}} \cdot \left(f(x)^{g(x)}\right)' = \frac{g(x)}{f(x)} \cdot f'(x) + g'(x) \cdot \ln(f(x)) \\ \Rightarrow\ & \qquad \left(f(x)^{g(x)}\right)' = \left(\frac{g(x)}{f(x)} \cdot f'(x) + g'(x) \cdot \ln(f(x))\right) \cdot f(x)^{g(x)}, \end{aligned}$$

which is a general formula for $f(x)^{g(x)}$ that we can calculate using only the derivatives of $f(x)$ and $g(x)$. Convenient, right?

To illustrate this once more, we study one last example:

Examples. Calculate

$$\left(\sin(x)^{\sin(x)}\right)'.$$

Proof. Simply plug in the formula we established above, for $f(x) = g(x) = \sin(x)$:

$$\begin{aligned}\left(\sin(x)^{\sin(x)}\right)' &= \left(\frac{g(x)}{f(x)} \cdot f'(x) + g'(x) \cdot \ln(f(x))\right) \cdot f(x)^{g(x)} \\ &= \left(\frac{\sin(x)}{\sin(x)} \cdot \cos(x) + \cos(x) \cdot \ln(\sin(x))\right) \cdot \sin(x)^{\sin(x)} \\ &= \sin(x)^{\sin(x)} \cos(x) \left(1 + \ln(\sin(x))\right).\end{aligned}$$

□

4 L'Hôpital's Rule

L'Hôpital's rule! It's your favorite rule! It's the following theorem:

Theorem. Suppose that $f(x)$ and $g(x)$ are a pair of differentiable functions such that $g'(x)$ is nonzero near some value a, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ and these limits are both 0 or both ∞, and the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Notice that we really need $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ to exist; if it does not, like in the example

$$\lim_{x\to\infty} \frac{x + \sin(x)}{x},$$

we have that

$$\lim_{x\to\infty} \frac{x + \sin(x)}{x} = \lim_{x\to\infty} 1 + \frac{\sin(x)}{x} = 1$$

while

$$\lim_{x\to\infty} \frac{\frac{d}{dx}(x + \sin(x))}{\frac{d}{dx}(x)},$$

To illustrate the use of this theorem, we calculate an example:

Examples. Show that

$$\lim_{x\to 0} \frac{\sin(x) - x + \frac{x^3}{3!}}{x^5} = \frac{1}{120}.$$

Proof. We repeatedly apply L'Hôpital's rule to this limit. This is justified because in each step the numerator and denominator are continuous functions (as they are sums of trig functions and polynomials) that are all 0 when we plug in $x = 0$, and because the final limit exists.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(x)-x+\frac{x^3}{3!}}{x^5} &\overset{L'H}{=} \lim_{x\to 0}\frac{\cos(x)-1+\frac{x^2}{2}}{5x^4}\\
&\overset{L'H}{=} \lim_{x\to 0}\frac{-\sin(x)+x}{20x^3}\\
&\overset{L'H}{=} \lim_{x\to 0}\frac{-\cos(x)+1}{60x^2}\\
&\overset{L'H}{=} \lim_{x\to 0}\frac{\sin(x)}{120x}\\
&\overset{L'H}{=} \lim_{x\to 0}\frac{\cos(x)}{120}\\
&=\frac{1}{120}.
\end{aligned}$$

□

5 The Integral Test: Statement

Finally, we discuss the integral test, which says the following:

Theorem. (Integral test:) If $f(x)$ is a positive and monotonically decreasing function, then

$$\sum_{n=N}^{\infty} f(n) \text{ converges if and only if } \int_N^{\infty} f(x)dx \text{ converges.}$$

It bears noting that the conditions "positive and monotonically decreasing" are *extremely necessary* in the above definition: if you examine functions that aren't monotonically decreasing, you can run into things like

$$f(x)=\begin{cases} x, & x\in\mathbb{N} \\ 0, & \text{otherwise}\end{cases}$$

which has the unfortunate property that

$$\sum_{n=N}^{\infty} f(n) = \sum_{n=N}^{\infty} n = \infty$$

while

$$\int_N^{\infty} f(x)dx = 0.$$

To illustrate this theorem's use, we turn to some examples:

6 The Integral Test: Examples

Question. Does the series

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

converge?

Proof. First, notice that because

$$\left(xe^{-x^2}\right)' = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} < 0, \forall x > 1,$$

we know that this function is decreasing on all of $[1,\infty)$. As well, it is trivially positive on $[1,\infty)$: so we can apply the integral test to see that this series converges iff the integral

$$\int_{n=1}^{\infty} xe^{-x^2}dx$$

converges.

But this is not too hard to show! – by using the u-substitution $u = x^2$, we have that

$$\int_{n=1}^{\infty} xe^{-x^2}dx = \int_{1}^{\infty} \frac{e^{-u}}{2}du = -\frac{e^{-u}}{2}\bigg|_1^{\infty} = \frac{e}{2},$$

and that in particular this integral converges. Therefore

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

must convege as well. □

Question. Does the series

$$\sum_{n=3}^{\infty} \frac{1}{(\ln(n))^{\ln(\ln(n))}}$$

converge?

Proof. First, notice that because $\ln(x)$ is an increasing function, so is $(\ln(x))^{\ln(\ln(x))}$; consequently, $\frac{1}{(\ln(x))^{\ln(\ln(x))}}$ is a decreasing function on $[3,\infty)$. As well, because $\ln(x)$ is positive whenever $x > 1$, we know that this function is positive on $[3,\infty)$: so we can apply the integral test to see that our sum converges iff the integral

$$\int_{n=3}^{\infty} \frac{1}{(\ln(x))^{\ln(\ln(x))}}dx$$

converges.

So: perform the u-substitution $u = \ln(x), x = e^u, dx = e^u du$ to see that

$$\begin{aligned}\int_{n=3}^{\infty} \frac{1}{(\ln(x))^{\ln(\ln(x))}} dx &= \int_{n=3}^{\infty} \frac{1}{u^{\ln(u)}} \cdot e^u du \\ &= \int_{n=3}^{\infty} \frac{1}{e^{(\ln(u))^2}} \cdot e^u du \\ &= \int_{n=3}^{\infty} e^{u-(\ln(u))^2} du.\end{aligned}$$

Finally, because

$$\lim_{u\to\infty} u - (\ln(u))^2 = \infty,$$

we know that $e^{u-(\ln(u))^2}$ increases without bound, and thus its integral cannot converge. So our sum does not converge, as well. □

Sometimes the integral test itself may not be applicable, but the idea of relating integrals and sums can still be used to show something converges! To see an example of this, look at our last problem:

Question. Does the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

converge?

Proof. No. To see why, simply notice that for any $k > 1 \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} &\geq \sum_{n=k}^{\infty} \frac{1}{n^{1+1/n}} \\ &\geq \sum_{n=k}^{\infty} \frac{1}{n^{1+1/k}}\end{aligned}$$

Now, notice that because $\frac{1}{x^{1+1/k}}$ is a decreasing function, we know that the sum $\sum_{n=k}^{\infty} \frac{1}{n^{1+1/k}}$ is strictly larger than the integral $\int_k^{\infty} \frac{1}{x^{1+1/k}} dx$; consequently, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}} \geq \int_{k}^{\infty}\frac{1}{x^{1+1/k}}dx \\ =&\frac{1}{x^{1/k}}\cdot(-k)\Bigg|_{k}^{\infty} \\ =&0-\frac{-k}{k^{1/k}} \\ =&k^{k-1/k} \\ \geq&\sqrt{k}, \forall k>2.\end{aligned}$$

Thus, we have that

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}} \geq \sqrt{k}, \forall k>2;$$

consequently, this sum must diverge. □