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Limits and Continuity

Week 4	*Caltech 2012*

1 Continuity: Definitions

Definition. If $f : X \to Y$ is a function between two subsets X, Y of $\mathbb{R}$, we say that

$$\lim_{x \to a} f(x) = L$$

if and only if

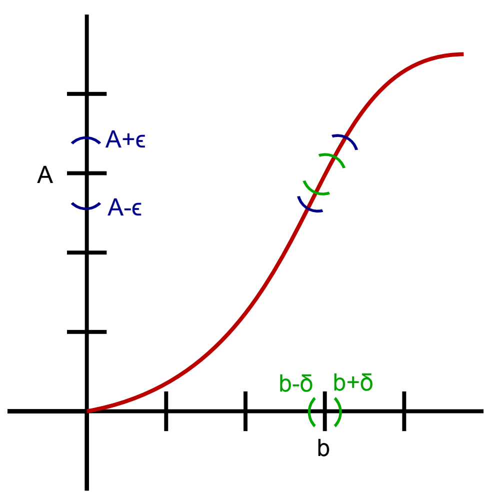
1. (vague:) as x approaches a, $f(x)$ approaches L.
2. (precise; wordy:) for any distance $\epsilon > 0$, there is some neighborhood $\delta > 0$ of a such that whenever $x \in X$ is within δ of a, $f(x)$ is within ϵ of L.
3. (precise; symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

Definition. A function $f : X \to Y$ is said to be **continuous** at some point $a \in X$ iff

$$\lim_{x \to a} f(x) = f(a).$$

Somewhat strange definitions, right? At least, the two "rigorous" definitions are somewhat strange: how do these epsilons and deltas connect with the rather simple concept of "as x approaches a, $f(x)$ approaches $f(a)$"? To see this a bit better, consider the following image:



This graph shows pictorially what's going on in our "rigorous" definition of limits and continuity: essentially, to rigorously say that "as x approaches a, $f(x)$ approaches $f(a)$", we are saying that

- for any distance ϵ around $f(a)$ that we'd like to keep our function,
- there is a neighborhood $(a-\delta, a+\delta)$ around a such that
- if f takes only values within this neighborhood $(a-\delta, a+\delta)$, it stays within ϵ of $f(a)$.

Basically, what this definition says is that if you pick values of x sufficiently close to a, the resulting $f(x)$'s will be as close as you want to be to $f(a)$ – i.e. that "as x approaches a, $f(x)$ approaches $f(a)$."

This, hopefully, illustrates what our definition is trying to capture – a concrete notion of something like convergence for functions, instead of sequences. So: how can we prove that a function f has some given limit L? Motivated by this analogy to sequences, we have the following blueprint for a proof-from-the-definitions that $\lim_{x \to a} f(x) = L$:

1. First, examine the quantity

$$|f(x) - L|.$$

   Using algebra/cleverness, try to find a simple upper bound for this quantity of the form

$$(\text{things bounded when } x \text{ is near } a) \cdot (\text{function based on} |x-a|).$$

   Some sample candidates: things like $|x-a| \cdot (\text{constants})$, or $|x-a|^3 \cdot (\text{bounded functions like } \sin(x))$.

2. Take your bounded part, and **bound** it! In other words, find a constant bound $C > 0$ and a value $\delta_1 > 0$ such that whenever x is within δ_1 of a, we have

$$(\text{bounded things}) < C.$$

3. Take your function based on $|x-a|$ and your constant C from the above step, and starting from the equation

$$(\text{function based on } |x-a|) < \frac{\epsilon}{C},$$

   solve for $|x-a|$ in terms of ϵ and C, by performing only reversible steps. This then gives you some equation of the form

$$|x-a| < (\text{thing in terms of } C, \epsilon\text{'s}).$$

   Define δ_2 to be this "thing in terms of C, ϵ's."

4. Let $\delta = \min(\delta_1, \delta_2)$. Then, whenever $|x-a| < \delta$, we have just proven that we satisfy both the equations

$$(\text{bounded things}) < C, \qquad \text{and}$$
$$(\text{function in } |x-a|) < \frac{\epsilon}{C}.$$

   If we combine these observations with the simple bound we derived in our first step, we've proven that whenever $|x-a| < \delta$, we have

$$|f(x) - L| < (\text{bounded things})(|x-a| \text{ things}) < C \cdot \frac{\epsilon}{C} = \epsilon.$$

   But this is exactly what we wanted to prove – this is the $\epsilon - \delta$ definiton of a limit! So we are done.

The following example ought to illustrate what we're talking about here:

2 Continuity: An Example

Claim 1. *The function $\frac{1}{x^2}$ is continuous at every point $a \neq 0$.*

Proof. We want to prove that $\lim_{x\to a} \frac{1}{x^2} = \frac{1}{a^2}$, for any $a \neq 0$.

We proceed according to our blueprint:

1. First, we examine the quantity $\left|\frac{1}{x^2} - \frac{1}{a^2}\right|$:

$$\begin{aligned}
\left|\frac{1}{x^2} - \frac{1}{a^2}\right| &= \left|\frac{a^2}{a^2x^2} - \frac{x^2}{a^2x^2}\right| \\
&= \left|\frac{a^2 - x^2}{a^2x^2}\right| \\
&= \left|\frac{(a-x)(a+x)}{a^2x^2}\right| \\
&= |a-x| \cdot \left|\frac{(a+x)}{a^2x^2}\right| \\
&= |x-a| \cdot \left|\frac{(a+x)}{a^2x^2}\right|.
\end{aligned}$$

   By algebraic simplification, we've broken our expression into two parts: one of which is $|x-a|$, and the other of which is bounded near $x = a$.

   For values of x rather close to a, because $a \neq 0$, we can bound this as follows: pick x such that x is within $a/2$ of a. Then we have

$$\begin{aligned}
\left|\frac{(a+x)}{a^2x^2}\right| &\leq \left|\frac{(a+(3a/2))}{a^2x^2}\right| \\
&\leq \left|\frac{(a+(3a/2))}{a^2(a/2)^2}\right| \\
&= \left|\frac{10}{a^3}\right|
\end{aligned}$$

$$,$$

   which is some nicely bounded constant. So, when we pick our δ, if we just make sure that $\delta < a/2$, we know that we have this quite simple and excellent upper bound

$$\left|\frac{(a+x)}{a^2x^2}\right| < \left|\frac{10}{a^3}\right|.$$

2. So: we have bounded the bounded part by $\left|\frac{10}{a^3}\right|$. Now, we want to take the remaining $|x-a|$ part, which is exactly $|x-a|$, and solve the equation

$$|x-a| < \frac{\epsilon}{10/a^3} = \frac{a^3\epsilon}{10}$$

   for $|x-a|$, given any arbitrary $\epsilon > 0$. Conveniently, this is already done! In fact, if we're using our blueprint and we can make our "function in terms of $|x-a|$" precisely

$|x-a|$, this is always this easy. Therefore, if we set $\delta_2 = \frac{a^3\epsilon}{10}$, then whenever $|x-a| < \delta_2$, we have

$$|x-a| < \frac{\epsilon}{10/a^3} = \frac{a^3\epsilon}{10}.$$

3. Now, set $\delta = \min(\delta_1, \delta_2)$. Then, whenever $|x-a| < \delta$, we have

$$|f(x) - L| < |x-a| \cdot \left|\frac{(a+x)}{a^2x^2}\right| < \frac{10}{a^3} \cdot \frac{a^3\epsilon}{10} = \epsilon,$$

which is precisely what we needed to show to satisfy the $\epsilon - \delta$ definition of a limit. Therefore, we have proven that $\lim_{x\to a} \frac{1}{x^2} = \frac{1}{a^2}$ for any $a \neq 0$, as claimed.

□

3 Continuity: Three Useful Tools

Limits and continuity are wonderfully useful concepts, but working with them straight from the definitions – as we saw above – can be somewhat ponderous. As a result, we have developed a number of useful tools and theorems to allow us to prove that certain limits exist without going through the definition every time: we present three such tools, and examples for each, here.

Theorem. (Squeeze theorem:) If f, g, h are functions defined on some interval $I \setminus \{a\}$[1] such that

$$f(x) \leq g(x) \leq h(x), \forall x \in I \setminus \{a\},$$
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x),$$

then $\lim_{x\to a} g(x)$ exists, and is equal to the other two limits $\lim_{x\to a} f(x), \lim_{x\to a} h(x)$.

Examples.

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

Proof. So: for all $x \in \mathbb{R}, x \neq 0$, we have that

$$-1 \leq \sin(1/x) \leq 1$$
$$\Rightarrow -x^2 \leq x^2 \sin(1/x) \leq x^2;$$

thus, by the squeeze theorem, as the limit as $x \to 0$ of both $-x^2$ and x^2 is 0,

$$\lim_{x\to 0} x^2 \sin(1/x) = 0$$

as well. □

[1]The set $X \setminus Y$ is simply the set formed by taking all of the elements in X that are not elements in Y. The symbol $\setminus$, in this context, is called "set-minus", and denotes the idea of "taking away" one set from another.

Theorem. (Limits and arithmetic): if f, g are a pair of functions such that $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ both exist, then we have the following equalities:

$$\begin{aligned}
\lim_{x\to a}(\alpha f(x) + \beta g(x)) &= \alpha\left(\lim_{x\to a} f(x)\right) + \beta\left(\lim_{x\to a} g(x)\right) \\
\lim_{x\to a}(f(x)\cdot g(x)) &= \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right) \\
\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) &= \left(\lim_{x\to a} f(x)\right)/\left(\lim_{x\to a} g(x)\right), \text{ if } \lim_{x\to a} g(x) \neq 0.
\end{aligned}$$

Corollary 2. *Every polynomial is continuous everywhere.*

Proof. To start, we know that the functions $f(x) = x$ and $f(x) = 1$ are trivially continuous. By multiplying these functions together and scaling by constant factors, we can create any polynomial; thus, by the above theorem, we know that any polynomial must be continuous, as we can create it from continuous things through arithmetical operations. □

Theorem. (Limits and composition): if $f : Y \to Z$ is a function such that $\lim_{y\to a} f(x) = L$, and $g : X \to Y$ is a function such that $\lim_{x\to b} g(x) = a$, then

$$\lim_{x\to b} f(g(x)) = L.$$

Specifically, if both functions are continuous, their composition is continuous.

Examples.

$$\lim_{x\to a} \sin(1/x^2) = \sin(1/a^2),$$

if $a \neq 0$.

Proof. By our work earlier in this lecture, $1/x^2$ is continuous at any value of $a \neq 0$, and from class $\sin(x)$ is continuous everywhere: thus, we have that their composition, $\sin(1/a^2)$, is continuous wherever $x \neq 0$. Thus,

$$\lim_{x\to a} \sin(1/x^2) = \sin(1/a^2),$$

as claimed. □

4 Discontinuity Proofs: A Lemma and a Blueprint

How do we show a function is discontinuous? Specifically: in our last class, we described a "blueprint" for showing that a given function was continuous at a point. Can we do the same for the concept of discontinuity?

As it turns out, we can! Specifically, we have the following remarkably useful lemma, proved in Dr. Ramakrishnan's class:

Lemma 3. *For any function $f : X \to Y$, we know that $\lim_{x\to a} f(x) \neq L$ iff there is some sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:*

- $\lim_{n\to\infty} a_n = L$, *and*

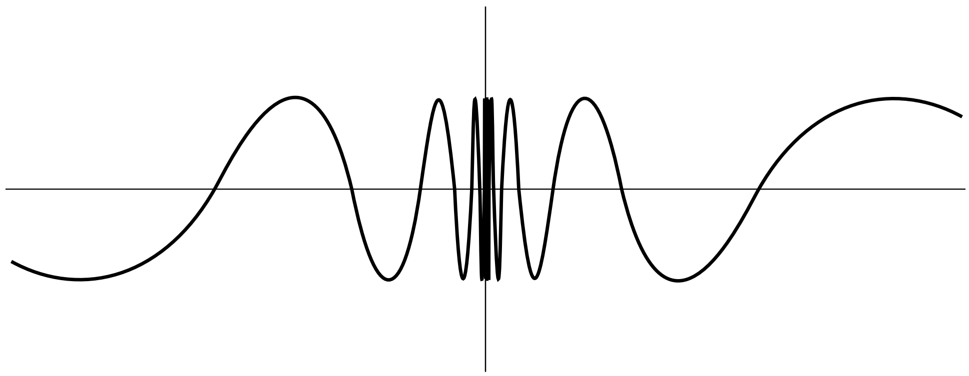
- $\lim_{n\to\infty} f(a_n) \neq L$, *and*

This lemma makes proving that a function f is discontinuous at some point a remarkably easy: all we have to do is find a sequence $\{a_n\}_{n=1}^{\infty}$ that converges to a on which the values $f(a_n)$ fail to converge to $f(a)$. Basically, it allows us to work in the world of sequences instead of that of continuity; a change that makes a lot of our calculations easier to make.

The following example should help illustrate our method:

Claim 4. *The function* $\sin(1/x)$ *has no defined limit at 0.*

Proof. So: before we start, consider the graph of $\sin(1/x)$:



Visual inspection of this graph makes it clear that $\sin(1/x)$ cannot have a limit as x approaches 0; but let's rigorously prove this using our lemma, so we have an idea of how to do this in general.

So: we know that $\sin\left(\frac{4k+1}{2}\pi\right) = 1$, for any k. Consequently, because the sequence $\left\{\frac{2}{(4k+1)\pi}\right\}_{k=1}^{\infty}$ satisfies the properties

- $\lim_{k\to\infty} \frac{2}{(4k+1)\pi} = 0$ and
- $\lim_{k\to\infty} \sin\left(\frac{1}{2/(4k+1)\pi}\right) = \lim_{k\to\infty} \sin\left(\frac{4k+1}{2}\pi\right) = \lim_{k\to\infty} 1 = 1$,

our lemma says that if $\sin(1/x)$ has a limit at 0, it must be 1.

However: we also know that $\sin\left(\frac{4k+3}{2}\pi\right) = -1$, for any k. Consequently, because the sequence $\left\{\frac{2}{(4k+3)\pi}\right\}_{k=1}^{\infty}$ satisfies the properties

- $\lim_{k\to\infty} \frac{2}{(4k+3)\pi} = 0$ and
- $\lim_{k\to\infty} \sin\left(\frac{1}{2/(4k+3)\pi}\right) = \lim_{k\to\infty} \sin\left(\frac{4k+3}{2}\pi\right) = \lim_{k\to\infty} -1 = -1$,

our lemma **also** says that if $\sin(1/x)$ has a limit at 0, it must be -1. Thus, because $-1 \neq 1$, we have that the limit $\lim_{x\to 0} \sin(1/x)$ cannot exist, as claimed.

□

5 One-Sided Limits

Let's conclude with something fairly elementary: the concept of a **one-sided limit**.

Definition. For a function $f : X \to Y$, we say that

$$\lim_{x \to a^+} f(x) = L$$

if and only if

1. (vague:) as x goes to a from the right-hand-side, $f(x)$ goes to L.
2. (concrete, symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta \text{ and } x > a) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x \to a^-} f(x) = L$$

if and only if

1. (vague:) as x goes to a from the left-hand-side, $f(x)$ goes to L.
2. (concrete, symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta \text{ and } x < a) \Rightarrow (|f(x) - L| < \epsilon).$$

Basically, this is just our original definition of a limit except we're only looking at x-values on one side of the limit point a: hence the name "one-sided limit." Thus, our methods for calculating these limits are pretty much identical to the methods we introduced on Monday: we work one example below, just to reinforce what we're doing here.

Claim 5.

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

Proof. First, examine the quantity

$$\frac{|x|}{x}.$$

For $x > 0$, we have that

$$\frac{|x|}{x} = 1;$$

therefore, for any $\epsilon > 0$, it doesn't even matter what δ we pick! – because for any x with $0 < x$, we have that

$$\left| \frac{|x|}{x} - 1 \right| = 0 < \epsilon.$$

Thus, the limit as $\frac{|x|}{x}$ approaches 0 from the right hand side is 1, as claimed. □

One-sided limits are particularly useful when we're discussing limits at infinity, as we describe in the next section:

6 Limits at Infinity

Definition. For a function $f : X \to Y$, we say that

$$\lim_{x\to+\infty} f(x) = L$$

if and only if

1. (vague:) as x goes to "infinity," $f(x)$ goes to L.
2. (concrete, symbols:)

$$\forall\epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x > N) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x\to-\infty} f(x) = L$$

if and only if

1. (vague:) as x goes to "negative infinity," $f(x)$ goes to L.
2. (concrete, symbols:)

$$\forall\epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x < N) \Rightarrow (|f(x) - L| < \epsilon).$$

In class, we described a rather useful trick for calculating limits at infinity:

Proposition. For any function $f : X \to Y$,

$$\lim_{x\to+\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right).$$

Similarly,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right).$$

The use of this theorem is that it translates limits at infinity (which can be somewhat complex to examine) into limits at 0, which can be in some sense a lot easier to deal with: as opposed to worrying about what a function does at extremely large values, we can just consider what a different function does at rather small values (which can make our lives often a lot easier.)

Here's an example, to illustrate where this comes in handy:

Claim 6.

$$\lim_{x\to+\infty} \frac{3x^2 + \cos(34x) + 10^7 \cdot x}{2x^2 + 1} = \frac{3}{2}.$$

Proof. Motivated by our proposition above, let us subsitute $1/x$ for x, so that we have

$$\lim_{x\to+\infty} \frac{3x^2+\cos(34x)+10^7\cdot x}{2x^2+1} = \lim_{x\to 0^+} \frac{3(1/x)^2+\cos(34/x)+10^7\cdot(1/x)}{2(1/x)^2+1}.$$

Multiplying both top and bottom by x^2, this limit is equal to

$$\lim_{x\to 0^+} \frac{3+x^2\cos(34/x)+10^7\cdot x}{2+x^2}.$$

Because limits play nicely with arithmetic, we know that the limit of this ratio is the ratio of the two limits $3+x^2\cos(34/x)+10^7\cdot x$ and $2+x^2$, **if and only iff both limits exist.**

But that's simple to see: because $2+x^2$ is a polynomial, it's continuous, and thus

$$\lim_{x\to 0^+} 2+x^2 = 2+0^2 = 2.$$

As well, because

$$3-x^2+1-^7\cdot x \le 3+x^2\cos(34/x)+10^7\cdot x \le 3+x^2+1-^7\cdot x,$$

and both of those polynomials converge to 3 as $x\to 0^+$, the squeeze theorem tells us that

$$\lim_{x\to 0^+} 3+x^2\cos(34/x)+10^7\cdot x = 3$$

as well.

Thus, because both limits exist, we have that

$$\lim_{x\to 0^+} \frac{3+x^2\cos(34/x)+10^7\cdot x}{2+x^2} = \frac{\lim_{x\to 0^+}(3+x^2\cos(34/x)+10^7\cdot x)}{\lim_{x\to 0^+}(2+x^2)} = \frac{3}{2},$$

as claimed. □

One useful application of limits at infinity comes through studying the intermediate value theorem, which is the subject of our next section:

7 The Intermediate Value Theorem

Theorem. If f is a continuous function on $[a,b]$, then f takes on every value between $f(a)$ and $f(b)$ at least once.

Most uses of this theorem occur when we have a continuous function f that takes on both positive and negative values on some interval; in this case, the intermediate value theorem tells us that this function must have a zero between each pair of sign changes. Basically, when you have a question that's asking you to find zeroes of a function, or to show that a function with prescribed endpoint behavior takes on some other values, the IVT is the way to go.

To illustrate this, consider the following example:

Claim 7. *If $p(x)$ is an odd-degree polynomial, it has a root in $\mathbb{R}$ – i.e. there is some $x\in\mathbb{R}$ such that $p(x)=0$.*

Proof. Write

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n,$$

where n is an odd natural number and $a_n > 0$. (The case where $a_n < 0$ is identical to the proof we're about to do if you flip all of the inequalities, so we omit it here by symmetry.)

Then, notice that

$$\begin{aligned}
\lim_{x\to+\infty} \frac{a_0 + \ldots + a_n x^n}{x^n} &= \lim_{x\to+\infty} \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots + \frac{a_{n-1}}{x} + a_n\right) \\
&= \lim_{x\to+\infty} \left(\frac{a_0}{x^n}\right) + \lim_{x\to+\infty} \left(\frac{a_1}{x^{n-1}}\right) + \ldots + \lim_{x\to+\infty} (a_n) \\
&= 0 + \ldots + 0 + a_n \\
&= a_n,
\end{aligned}$$

(where the second line is justified because all of the individual limits exist.)

As a result, we know that for large positive values of x, $\frac{a_0+\ldots+a_n x^n}{x^n}$ is as close to a_n as we would like. Specifically, we know that for large values of x, we have that the distance between $\frac{a_0+\ldots+a_n x^n}{x^n}$ and a_n is less than, say, $a_n/2$. As a consequence, we have specifically that $\frac{a_0+\ldots+a_n x^n}{x^n}$ is **positive**, for large positive values of x – thus, for some large positive x, we have that

$$x^n \cdot \frac{a_0 + \ldots + a_n x^n}{x^n} = (\text{positive}) \cdot (\text{positive}) = (\text{positive}).$$

Similarly, because

$$\begin{aligned}
\lim_{x\to-\infty} \frac{a_0 + \ldots + a_n x^n}{x^n} &= \lim_{x\to-\infty} \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots + \frac{a_{n-1}}{x} + a_n\right) \\
&= \lim_{x\to-\infty} \left(\frac{a_0}{x^n}\right) + \lim_{x\to-\infty} \left(\frac{a_1}{x^{n-1}}\right) + \ldots + \lim_{x\to-\infty} (a_n) \\
&= 0 + \ldots + 0 + a_n \\
&= a_n,
\end{aligned}$$

we also have that for large **negative** values of x, $\frac{a_0+\ldots+a_n x^n}{x^n}$ is as close to a_n as we'd like, and thus that $\frac{a_0+\ldots+a_n x^n}{x^n}$ is **positive**, for large **negative** values of x. Thus, for some large negative value of x, we have that

$$x^n \cdot \frac{a_0 + \ldots + a_n x^n}{x^n} = (\text{negative}) \cdot (\text{positive}) = (\text{negative}).$$

(Notice that the fact that n was odd was used in the above calculation, to insure that x negative implies that x^n is negative.)

We have thus shown that our polynomial adopts at least one positive and one negative value: thus, by the intermediate value theorem, it must be 0 somewhere between these two values! Thus, our polynomial has a root, as claimed. $\square$

8 Open, Closed, and Bounded Sets

Finally, we make something of a detour here, to quickly define open, closed, and bounded sets:

Definition. A set $X \subset \mathbb{R}$ is called **open** if for any $x \in X$, there is some neighborhood δ_x of x such that the entire interval $(x - \delta_x, x + \delta_x)$ lies in X.

Examples.

- The sets $\mathbb{R}$ and $\emptyset$ are both trivially open sets.
- Any open interval (a, b) is an open set.
- The union[2] of arbitrarily many open sets is open.
- The intersection [3] of finitely many open sets is open.

Definition. A set $X \subset \mathbb{R}$ is called **closed** if its complement[4] is open.

Examples.

- The sets $\mathbb{R}$ and $\emptyset$ are both trivially closed sets. Note that this means that some sets can be both open and closed!
- Any closed interval $[a, b]$ is an closed set.
- The intersection of arbitarily many closed sets is closed.
- The union of finitely many closed sets is closed.

Definition. A set $X \subset \mathbb{R}$ is bounded iff there is some value $M \in \mathbb{R}$ such that $-M \leq x \leq M$, for any $x \in X$.

We will work more closely with these definitions in future lectures: however, for now, it suffices to note the following useful theorem, which we'll use heavily in our discussion of the derivative:

Theorem. (Extremal value theorem:) If $f : X \to Y$ is a continuous function, and X is a closed and bounded subset X of $\mathbb{R}$, then f attains its minima and maxima. In other words, there are values $m, M \in X$ such that for any $x \in X$, $f(m) \leq f(x) \leq f(M)$.

[2]The union $X \cup Y$ of two sets X, Y is the set $\{a : a \in X$ or $a \in Y$, or both.$\}$
[3]The intersection $X \cap Y$ of two sets X, Y is the set $\{a : a \in X$ and $a \in Y.\}$
[4]The complement X^c of a set X is the set $\{a : a \notin X\}$