Math 8	Instructor: Padraic Bartlett

Problems from Recitation

Week 5	*Caltech 2012*

Some people asked for writeups of the problems from rec. Here they are!

Question. Suppose that $a_1, \ldots a_n$ are all real numbers from the interval $[0,1]$. Show that

$$\prod_{i=1}^{n}(1-a_i) \geq 1 - \sum_{i=1}^{n} a_i.$$

Proof. We proceed by induction. Our base case is trivial: for $n = 1$, our claim is

$$\prod_{i=1}^{1}(1-a_i) = 1 - a_1 \geq 1 - a_1 = 1 - \sum_{i=1}^{1} a_i,$$

which is true.

Inductive step: assume that our claim holds for n. We seek to prove our claim for $n+1$: i.e. take any collection of $n+1$ values $a_1, \ldots a_{n+1}$. Examine the product

$$\prod_{i=1}^{n+1}(1-a_i) = (1-a_{n+1}) \cdot \left(\prod_{i=1}^{n}(1-a_i)\right).$$

If we apply our inductive hypothesis to $(\prod_{i=1}^{n}(1-a_i))$, we can see that

$$\begin{aligned}
\prod_{i=1}^{n+1}(1-a_i) &= (1-a_{n+1}) \cdot \left(\prod_{i=1}^{n}(1-a_i)\right) \\
&\geq (1-a_{n+1}) \cdot \left(1 - \sum_{i=1}^{n} a_i\right) \\
&= 1 - a_{n+1} - \left(\sum_{i=1}^{n} a_i\right) + a_{n+1} \cdot \left(\sum_{i=1}^{n} a_i\right) \\
&\geq 1 - a_{n+1} - \left(\sum_{i=1}^{n} a_i\right) \\
&= 1 - \sum_{i=1}^{n+1} a_i.
\end{aligned}$$

Notice that we used that all of the a_i's were ≤ 1 when we plugged in our inductive hypothesis (as otherwise $(1 - a_{n+1})$ would have been negative, and our inequality would have gotten reversed.) Also notice that we used that all of the $a_i's$ were positive when we said that $a_{n+1} \cdot (\sum_{i=1}^{n} a_i) \geq 0$ and therefore getting rid of it only makes the right-hand-side smaller. $\square$

Example. Show that $\sqrt[5]{39}$ is irrational.

Proof. We proceed by contradiction. Assume that it is rational: i.e. that there are integers p, q, relatively prime, q nonzero, such that

$$\sqrt[5]{39} = \frac{p}{q}.$$

Taking fifth powers and multiplying by q^5 gives us

$$39q^5 = p^5.$$

Because 3 is a factor of $39q^5$, it must be a factor of the right-hand-side (i.e. p^5) as well. Therefore, 3 must in fact be a factor of p, because 3 is prime. (This can be seen by factoring p into its unique prime factorization, and seeing that if there are no 3's in this factorization, then there can be no 3's in the factorization of p^5.)

Therefore, because 3 is a factor of p, 3^5 is a factor of p^5; i.e. we can write p^5 as $3^5 \cdot k$, for some other integer k. This means that we have

$$39q^5 = 3^5 k \Rightarrow \qquad 13q^5 = 3^4 k,$$

and therefore 3 is a factor of $13q^5$. In particular, because 13 is prime, this means that 3 is a factor of p^5 and therefore 3 is a factor of p. But this contradicts our claim that p and q have no common factors! □

Question. Let x, y be a pair of positive real numbers such that $x < y$. Show that

$$\lim_{n\to\infty} (x^n + y^n)^{1/n} = y.$$

(Hint: squeeze!

Proof. As suggested by the hint, we want to apply the squeeze theorem. Specifically, notice that

$$y = (y^n)^{1/n} < (x^n + y^n)^{1/n} < (y^n + y^n)^{1/n} = 2^{1/n} y.$$

Therefore, because

$$\lim_{n\to\infty} y = y$$

and

$$\lim_{n\to\infty} 2^{1/n} y = y \cdot \lim_{n\to\infty} 2^{1/n} = y \cdot 1 = y,$$

the squeeze theorem tells us that

$$\lim_{n\to\infty} (x^n + y^n)^{1/n} = y$$

as well. □

Example. Let $\alpha(n)$ denote the function that takes in any natural number n and returns the number of prime factors it has, counting repeated primes. For example, $\alpha(6) = 2, \alpha(8) = 3, \alpha(120) = 5, \alpha(17) = 1$. Let $\alpha(0) = 0$, to avoid any possibly ambiguous cases. Show that the series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^3}$$

converges.

Proof. To do this, simply notice that $\alpha(n) < n$ for any n, as the number of prime factors in a number can never exceed that number. (Why? Well, suppose that you have a number with k prime factors. Then this number is at least 2^k, because each prime factor contributes a multiplier of at least 2, the smallest prime. We know that $k < 2^k$, for all k, so this proves our claim.)

Using this, notice that we have

$$\frac{\alpha(n)}{n^3} < \frac{n}{n^3} = \frac{1}{n^2},$$

for all $n \in \mathbb{N}$: applying the comparison theorem then tells us that because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so must our series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^3}.$$

$\square$

Question. Let $\Delta(x)$ be the function that is 1 whenever $x > 1$, and 0 otherwise. Let $f(x)$ be the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\Delta(n^2x^2)}{n^4}.$$

Is $f(x)$ defined everywhere: i.e. for any x, does the series given by $f(x)$ converge? Is there a point where it is discontinuous? How about one where it is continuous?

Proof. Checking whether $f(x)$ is defined everywhere is relatively easy: we just use the comparison test! In particular, just notice that

$$\frac{\Delta(n^2x^2)}{n^4} \leq \frac{1}{n^4}$$

for any n, x. Therefore, because

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

converges, the series given by $f(x)$ must also converge.

We claim that $f(x)$ is discontinuous at 0. To see this, simply notice that for any $n, x > 1$, we have nx^2 greater than 1; therefore $\frac{\Delta(nx^2)}{n^4} = \frac{1}{n^4}$, for any $x > 1$, and

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

whenever $x > 1$.

However, for any $\frac{1}{2} < x < 1$, notice that $\Delta(1 \cdot x^2)$ is 0, while $\Delta(nx^2) = 1$ for any $n > 1$. Therefore, $f(x)$ is equal to

$$\sum_{n=2}^{\infty} \frac{1}{n^4}.$$

So, if we approach x from the right, we have that the limit

$$\lim_{x \to 1^+} f(x) = \sum_{n=1}^{\infty} \frac{1}{n^4},$$

because the function is constant and equal to $\sum_{n=1}^{\infty} \frac{1}{n^4}$ when $x > 1$. Conversely, if we approach x from the left, we have that

$$\lim_{x \to 1^-} f(x) = \sum_{n=2}^{\infty} \frac{1}{n^4},$$

because this function is also constant and equal to $\sum_{n=2}^{\infty} \frac{1}{n^4}$ on $(1/2, 1)$.

These limits are different; the right-side is a sum starting at $n = 1$, while the left-side starts at $n = 2$! Therefore, the function is discontinuous at $x = 1$.

We finally claim that $f(x)$ is continuous at 0: i.e. that

$$\lim_{x \to 0} f(x) = f(0) = 0.$$

To do this requires some work; while we may want to use the squeeze theorem, we can't really squeeze the terms of $f(x)$ piece-by-piece, because they all look like $\frac{1}{n^4}$'s until they become 0 when x gets sufficiently small.

Instead, the only technique we have is a fairly fundamental way of working with series-functions: breaking $f(x)$ into two portions, its **head** and its **tail**.

What do we mean by this? Well: take our series

$$\sum_{n=1}^{\infty} \frac{\Delta(n^2x^2)}{n^4}.$$

Break it into two parts: the **head** sum, from 1 to N, and the **tail** sum, from N to ∞:

$$\sum_{n=1}^{N} \frac{\Delta(n^2x^2)}{n^4} + \sum_{n=N+1}^{\infty} \frac{\Delta(n^2x^2)}{n^4}$$

Our key observations, and ones that typically hold for **any** convergent series-function-thing, are the following:

1. The **head** portion of our series, because it's the sum of a finite number of terms, is usually going to be **ridiculously nice** to work with: i.e. because it is the sum of a finite number of things, properties about these things (continuity, differentiability, knowing how to manipulate them) are easily preserved.

2. The **tail** portion of our series, because it starts at some very large N, is usually going to be **very small**. So, while nice properties might not survive (i.e. adding infinitely many continuous things often results in not-continuous things), the result will be so small that we won't care!

In practice, for our example, notice the following:

1. If $|x| < \frac{1}{N}$, then for any $n \leq N$, we have $n^2x^2 < 1$, and therefore that

$$\sum_{n=1}^{N} \frac{\Delta(n^2x^2)}{n^4} = \sum_{n=1}^{N} \frac{0}{n^4} = 0.$$

   So near 0, the head portion of our sum **is** 0, and is therefore something we don't have to worry about.

2. Near 0, the tail portion is harder to understand. However, we don't care, because it's **tiny**! In particular, notice that because

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

   converges, the limit

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{1}{n^4} = 0;$$

   this is because if a series of positive terms converges, then the "tail" of the series has to get arbitrarily small as we start at larger and larger values of N. (To see why, simply notice that if some sum converges, then by adding finitely many terms of the sum we can get as "close" to this sum as we want. This is the same as making the sum of the remaining terms as small as we want.)

   To be precise: for any $\epsilon > 0$, we can find a N such that the sum

$$\sum_{n=N}^{\infty} \frac{1}{n^4} < \epsilon.$$

So: combine these results! In other words, take any $\epsilon > 0$,and pick a N such that our "tail" is small: i.e. such that

$$\sum_{n=N}^{\infty} \frac{1}{n^4} < \epsilon.$$

Now pick any $|x| < \frac{1}{N}$; then, as noted above, our head portion

$$\sum_{n=1}^{N} \frac{\Delta(n^2x^2)}{n^4} = 0$$

for such a value of x.

Therefore, we've just shown that for any $\epsilon > 0$ and any $|x| < \frac{1}{N}$,

$$|f(x) - f(0)| = \left| \sum_{n=1}^{N} \frac{\Delta(n^2x^2)}{n^4} + \sum_{n=N+1}^{\infty} \frac{\Delta(n^2x^2)}{n^4} \right| < |0 + \epsilon| = \epsilon.$$

But this is just the $\epsilon - \delta$ definition of continuity! I.e. we've shown that for any $\epsilon > 0$, we can pick $\delta = \frac{1}{N}$ such that if $|x - 0| = |x| < \frac{1}{N}$, we have

$$|f(x) - f(0)| < \epsilon.$$

So it's continuous at 0!

□

Question. Let

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

Is $f(x)$ differentiable at 0?

Proof. We simply use the definition of the derivative: i.e. $f(x)$ is differentiable at 0 if and only if the limit

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

exists. Because $f(0) = 0$, this is just the limit

$$\lim_{h \to 0} \frac{f(h)}{h}.$$

Dealing with this directly is hard. However, for any h, because $|f(h)| \leq h^2$, we can bound this quantity above by $\frac{h^2}{|h|}$ and below by $-\frac{h^2}{|h|}$. We can see that both of these quantities (after simplifying them to $\pm|x|$) go to 0 as x goes to 0; therefore, using the squeeze theorem, we can finally note that our limit

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

as well. Therefore the derivative exists (because this limit exists!), and is specifically 0. □