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Lecture 9: The Integral Test

Week 9 *Caltech - Fall, 2011*

1 Random Questions

Question 1.1. You and 99 of your friends have just been teleported back in time to Soviet Russia, whereupon the KGB has promptly arrested all of you. You are all to be placed in separate cells in a prison in the middle of Siberia; there are no other prisoners in your prison, nor any guards other than the prison's warden, nor any way to escape or communicate with anyone in any other cells.

In fact, the only thing that will happen in your lives that involves interaction with other people is the following: At random intervals during the day, the warden will call up a random prisoner into their office. There are only two things in the office which you, as a prisoner, can touch or get at:

- A lightswitch. It doesn't seem to control anything, but you can flick it either up or down, and no one but the prisoners is able to touch it.
- A large, red button. If you press the red button before every prisoner has been in to see the warden at least once, the entire prison explodes and you all die (so don't do that.) However, if you press the red button after every prisoner has seen the warden at least once, then you all get to go free!

Before you arrive at the prison, your KGB handlers explain this situation to you all as a group, and leave you alone to come up with a plan. Can you collectively come up with a way to ensure that, given enough time, you'll always escape from the prison?

A slightly easier question: if you like, you can assume that your KGB handlers mention that the light switch starts in the up position. This isn't necessary, but it helps. A bit.

2 The Integral Test: Statement

This week's notes are shorter than normal, because of the Thanksgiving break! In them, we discuss the integral test, which is the following claim:

Theorem 2.1. (Integral test:) If $f(x)$ is a function that is eventually[1] monotonically decreasing, then

$$\sum_{n=N}^{\infty} f(n) \text{ converges if and only if } \int_N^{\infty} f(x)dx \text{ exists and is finite.}$$

[1] A function is **eventually** monotonically decreasing if and only if there is some cutoff value N past which it is monotonically decreasing.

Note that the condition "eventually monotonically decreasing" is *extremely necessary* in the above theorem: if you examine functions that are never monotonically decreasing, you can run into things like

$$f(x) = \begin{cases} x, & x \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

which has the unfortunate property that

$$\sum_{n=N}^{\infty} f(n) = \sum_{n=N}^{\infty} n = \infty,$$

while

$$\int_{N}^{\infty} f(x)dx = 0.$$

To illustrate this theorem's use, we calculate a few examples:

3 The Integral Test: Examples

Question 3.1. For what values of p does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converge?

Proof. Earlier in our course, we proved that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges; shortly afterwards, we proved that the series of reciprocals of squares,

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

converges.

So, by the comparison test, we know that for all $p \leq 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, and for all $p \geq 2$, this series converges. What about the values in between?

To study this, we use the integral test! First, notice that for any $p \in (1, 2)$, the function

$$f(x) = \frac{1}{x^p}, f : (0, \infty) \to (0, \infty)$$

is monotonically decreasing, because x^p is monotonically increasing and $f(x)$ is just the reciprocal of x^p. Therefore, by the integral test, we know that our series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ will converge precisely when the integral

$$\int_{1}^{\infty} \frac{1}{x^p} dx$$

exists and is finite.

We can calculate this integral easily, by just using the power rule $\frac{d}{dx}x^{1-p} = (1-p)x^{-p}$, which is valid for all $p \neq 1$, and integrating by using the antiderivative:

$$\int_1^\infty \frac{1}{x^p} dx = \left.\frac{x^{1-p}}{1-p}\right|_1^\infty = \left(\lim_{x\to\infty} \frac{x^{1-p}}{1-p}\right) - \frac{1^{1-p}}{1-p}$$
$$= \left(\lim_{x\to\infty} \frac{1}{x^{p-1}(1-p)}\right) + \frac{1}{p-1}$$
$$= 0 + \frac{1}{p-1}.$$

This integral exists and is finite for any $p > 1$; therefore, by the integral test and our earlier work with the comparison test, we've proven that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. $\square$

So: we know that $\sum \frac{1}{n}$ diverges, while $\sum \frac{1}{n^p}$ converges for any $p > 1$. Are there any series we can fit somehow "between" these two results: i.e. are there any series that are termwise smaller than the harmonic series's terms, but termwise eventually greater than the terms of any series $\sum \frac{1}{n^p}$, for any fixed p?

As it turns out, there are! Consider the following problem:

Question 3.2. Does the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

converge?

Proof. At first, it's hard to tell whether this series should converge or diverge: it's growing slower than the harmonic series, but eventually growing faster than any of the series $\sum \frac{1}{n^p}$. To find out what's happening, let's try to apply the integral test!

First, to apply the integral test, we need to show that the function

$$f(x) = \frac{1}{x^{1+\frac{1}{x}}}$$

is monotonically decreasing, or (equivalently) that the function

$$g(x) = x^{1+\frac{1}{x}}$$

is monotonically increasing. We know that a differentiable function is monotonically increasing if and only if its derivative is always positive; so it suffices to study the derivative of $g(x)$.

...How do we do that? At first glance, $g(x)$ doesn't look like any function we've dealt with before: it has variables both in the body of our function and the exponent, and doesn't immediately look like anything we've differentiated before. However, using the logarithm, we can transform it into a rather familiar form: first, recall that $x = e^{\ln(x)}$. Using this, we can write

$$g(x) = x^{1+\frac{1}{x}} = \left(e^{\ln(x)}\right)^{1+\frac{1}{x}} = e^{\ln(x)\cdot(1+\frac{1}{x})},$$

which we *do* know how to differentiate! Just use the chain rule and use that $\frac{d}{dx}e^x = e^x$:

$$\begin{aligned}
\frac{d}{dx}(g(x)) &= \frac{d}{dx}\left(e^{\ln(x)\cdot(1+\frac{1}{x})}\right) \\
&= \left(e^{\ln(x)\cdot(1+\frac{1}{x})}\right)\cdot\frac{d}{dx}\left(\ln(x)\cdot\left(1+\frac{1}{x}\right)\right) \\
&= \left(e^{\ln(x)\cdot(1+\frac{1}{x})}\right)\cdot\left(\frac{1}{x}\cdot\left(1+\frac{1}{x}\right)+\ln(x)\cdot\frac{-1}{x^2}\right) \\
&= \left(e^{\ln(x)\cdot(1+\frac{1}{x})}\right)\cdot\left(\frac{x+1-\ln(x)}{x^2}\right) \\
&= x^{1+\frac{1}{x}}\cdot\left(\frac{x+1-\ln(x)}{x^2}\right).
\end{aligned}$$

For any positive value of x, we know that $x^{1+\frac{1}{x}}$ is positive; as well, because $x > \ln(x)$, we know that $\frac{x+1-\ln(x)}{x^2}$ is also always positive. Therefore, we have that $g'(x) > 0$ on $(0, \infty)$, and therefore that $f(x)$ is a monotonically decreasing function.

Therefore, we can apply the integral test to our series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}},$$

and examine the integral

$$\int_{n=1}^{\infty} \frac{1}{x^{1+1/x}} dx.$$

Unfortunately, this integral is a bit of a pain to directly deal with: like the Gaussian, it's an integral that has no elementary antiderivative, so we can't just find an antiderivative for it.

However, we don't need to! To see why, simply notice that for any $k > 1 \in \mathbb{N}$,

$$\begin{aligned}
\int_{n=1}^{\infty} \frac{1}{x^{1+1/x}} dx &> \int_{n=k}^{\infty} \frac{1}{x^{1+1/x}} dx \\
&= \int_{n=k}^{\infty} \frac{1}{x^{1+1/k}} dx \\
&= \frac{1}{1-(1+1/k)} \cdot x^{1-(1+1/k)} \Big|_k^{\infty} \\
&= \frac{1}{-1/k)} \cdot x^{-1/k} \Big|_k^{\infty} \\
&= -\frac{k}{x^{1/k}} \Big|_k^{\infty} \\
&= \left(\lim_{x \to \infty} -\frac{k}{x^{1/k}} \right) + \frac{k}{k^{1/k}} \\
&= 0 + \frac{k}{k^{1/k}} \\
&= k^{1-1/k} \\
&> k^{1-1/2} = \sqrt{k}.
\end{aligned}$$

So, while we may not have calculated the integral $\int_{n=1}^{\infty} \frac{1}{x^{1+1/x}} dx$, we **have** shown that it's greater than $\sqrt{k}$, for any $k > 0$: in other words, we've shown that this integral has to be infinte! Therefore, by the integral test, our series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

must diverge as well. $\square$