Math 8 — Instructor: Padraic Bartlett

Lecture 3: Sequences and Series

Week 3 — *Caltech - Fall, 2011*

1 Random Questions

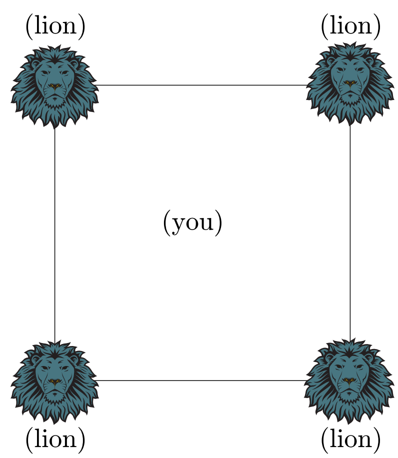
Question 1.1. *You've been teleported back in time to ancient Rome, where you've been thrown into a gladiatorial pit. The pit is a perfect square, which you're in the exact center of: furthermore, at each of the four vertices of the square, there is a lion.*

The lions are restrained by chains so that they can only walk along the perimeter of the square: as well, both you and the lions

- *run at the same speed,*
- *are point-masses that can change direction arbitrarily quickly,*
- *are arbitrarily brilliant and can execute any given strategy, and*
- *know no fear.*

The lions eat you if and only if they occupy the same point on the boundary of the square at the exact instant that you occupy that point on the boundary of the square.

Can you escape the square – i.e. is there a strategy you can adopt that will always allow you to run out of the square without being devoured? What about other regular polygons: which of those can you escape? How?



Question 1.2. *Take the question above, and suppose that instead you've been thrown into a pit that's a perfect circle. Suppose furthermore that the Romans, distrustful of the circle and its lack of flat sides, have placed* ***countably many lions*** *on the perimeter of your circle. Can you escape?*

Question 1.3. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive integers with the following properties:*

1. *The sums $\sum_{n=1}^{\infty} \frac{1}{a_n}$ and $\sum_{n=1}^{\infty} \frac{1}{b_n}$ both diverge.*
2. *$a_n < a_{n+1}, b_n < b_{n+1}$, for every $n \in \mathbb{N}$.*

Does the sum

$$\sum_{n=1}^{\infty} \frac{1}{a_n + b_n}$$

also have to diverge?

2 Sequences and Series

This week, we're going to discuss **sequences** and **series**. We start by making some basic definitions (like, say, what a sequence *is*) in the following section:

2.1 Sequences: Definitions and Tools

Definition 2.1. A **sequence** is a collection of objects (typically, numbers) $\{a_n\}_{n=1}^{\infty}$ indexed by the natural numbers.

Definition 2.2. A sequence $\{a_n\}_{n=1}^{\infty}$ is called **bounded** if there is some value $b \in \mathbb{R}$ such that $|a_n| < b$, for every $n \in \mathbb{N}$.

Definition 2.3. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **monotonically increasing** if $a_n \leq a_{n+1}$, for every $n \in \mathbb{N}$; conversely, a sequence is called**monotonically decreasing** if $a_n \geq a_{n+1}$, for every $n \in \mathbb{N}$.

Definition 2.4. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value λ if the a_n's "go to λ" at infinity. To put it more formally, $\lim_{n\to\infty} a_n = \lambda$ iff for any distance ϵ, there is some cutoff point N such that for any n greater than this cutoff point, a_n must be within ϵ of our limit λ.

In symbols:

$$\lim_{n\to\infty} a_n = \lambda \text{ iff } (\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

Convergence is one of the most useful properties of sequences; if you know that a sequence converges to some value λ, you in a sense know where the sequence is "going," and furthermore know where almost all of its value are going to be (close to λ.) Because convergence is so useful, we've developed a number of tools for determining where a sequence is converging to:

1. **The definition of convergence**: The simplest way to show that a sequence converges is sometimes just to use the definition of convergence. In other words, you want to show that for any distance ϵ, you can eventually force the a_n's to be within ϵ of our limit, for n sufficiently large.

   How can we do this? One method I'm fond of is the following approach:

- First, examine the quantity $|a_n - L|$, and try to come up with a very simple upper bound that depends on n and goes to zero. Example bounds we'd love to run into: $1/n, 1/n^2, 1/\log(\log(n))$.
- Using this simple upper bound, given $\epsilon > 0$, determine a value of N such that whenever $n > N$, our simple bound is less than ϵ.
- Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $n > N$, $|a_n - L| < \epsilon$.

2. **Arithmetic and sequences**: These tools let you combine previously-studied results to get new ones. Specifically, we have the following results:

   - *Additivity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$.
   - *Multiplicativity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
   - *Quotients of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$.

3. **Monotone and bounded sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded above and nonincreasing, it also converges. If a sequence is monotone, this is usually the easiest way to prove that your sequence converges, as both monotone and bounded are "easy" properties to work with. One interesting facet of this property is that it can tell you that a sequence converges without necessarily telling you what it converges to! So, it's often of particular use in situations where you just want to show something converges, but don't actually know where it converges *to*.

4. **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l. This is particularly useful for sequences with things like sin(horrible things) in them, as it allows you to "ignore" bounded bits that aren't changing where the sequence goes.

5. **Cauchy sequences**: We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m > n \geq N$, we have

   $$|a_m - a_n| < \epsilon.$$

   You can think of this condition as saying that Cauchy sequences "settle down" in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

   The Cauchy theorem, in this situation, is the following: a sequence is Cauchy if and only if it converges.

   The Cauchy criterion doesn't come up as often as the others in Math 1a (later in mathematics, however, it shows up pretty much everywhere!) Its main uses are for working with series (we'll have an example of this later, and define series later as well!), and for sequences whose limits we don't know: like the monotone-bounded-convergence theorem, this result doesn't need you to know where a sequence is converging to in order to show that it converges.

2.2 Sequences: Applications of Convergence Tools

In this section, we work an example for each of these tools. We start by illustrating how to prove a sequence converges using just the definition:

Claim 1. (Definition of convergence example:)

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0.$$

Proof. When we discussed the definition as a convergence tool, we talked about a "blueprint" for how to go about proving convergence from the definition: start with $|a_n - L|$, try to find a simple upper bound on this quantity depending on n, and use this simple bound to find for any ϵ a N such that $n > N$ implies that $|a_n - L| <$ (simple upper bound) $< \epsilon$. Let's try this! Specifically, examine the quantity $|\sqrt{n+1} - \sqrt{n} - 0|$:

$$\begin{aligned} |\sqrt{n+1} - \sqrt{n} - 0| &= \sqrt{n+1} - \sqrt{n} \\ &= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} - \sqrt{n}} \\ &= \frac{n+1-n}{\sqrt{n+1} - \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} - \sqrt{n}} \\ &< \frac{1}{\sqrt{n}}. \end{aligned}$$

All we did here was hit our $|a_n - L|$ quantity with a ton of random algebra, and kept trying things until we got something simple. The specifics aren't as important as the idea here: just start with the $|a_n - L|$ bit, and try everything until it's bounded by something simple and small!

In our specific case, we've acquired the upper bound $\frac{1}{\sqrt{n}}$, which looks rather simple: so let's see if we can use it to find a value of N.

Take any $\epsilon < 0$. If we want to make $\frac{1}{\sqrt{n}} < \epsilon$, we merely need to pick N such that $\frac{1}{\sqrt{N}} < \epsilon$, and then select $n > N$.

This then tells us that for any $\epsilon > 0$, we can find a N such that for any $n > N$, we have

$$|\sqrt{n+1} - \sqrt{n} - 0| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon,$$

which is the definition of convergence. So we've proven that $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$. ☐

Claim 2. (Arithmetic and Sequences example:) The sequence

$$a_1 = 1,$$
$$a_{n+1} = \sqrt{1 + a_n^2}$$

does not converge.

Proof. We proceed by contradiction: in other words, suppose that this sequence does converge to some value L, say. Then, examine the limit

$$\lim_{n\to\infty} a_n^2.$$

Because convergent sequences are multiplicative, we know that

$$\lim_{n\to\infty} a_n^2 = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} a_n) = L \cdot L = L^2.$$

However, we can also use the recursive definition of the a_n's to see that

$$\begin{aligned}\lim_{n\to\infty} a_n^2 &= \lim_{n\to\infty} \left(\sqrt{1 + a_{n-1}^2}\right)^2 \\ &= \lim_{n\to\infty} (1 + a_{n-1}^2) \\ &= (\lim_{n\to\infty} 1) + (\lim_{n\to\infty} a_{n-1}^2) \\ &= 1 + (\lim_{n\to\infty} a_{n-1}^2)\end{aligned}$$

However, we know that $\lim_{n\to\infty} a_{n-1}^2 = \lim_{n\to\infty} a_n^2$, because the two sequences are the same (just shifted over one place) and thus have the same behavior at infinity. So we have in fact that

$$\lim_{n\to\infty} a_n^2 = 1 + (\lim_{n\to\infty} a_{n-1}^2) = 1 + (\lim_{n\to\infty} a_n^2) = 1 + (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} a_n) = 1 + L^2,$$

and thus that $L^2 = 1 + L^2$: i.e. $0 = 1$. This is clearly nonsense: so we've arrived at a contradiction. Therefore, our original assumption (that our sequence $\{a_n\}_{n=1}^{\infty}$ converged must be false – i.e. this sequence must diverge, as claimed. $\square$

Claim 3. (Monotone convergence theorem example:)

$$\lim_{n\to\infty} 2^{1/n} = 1.$$

Proof. Let's start by using the monotone-bounded convergence theorem to show that the sequence $\{2^{1/n}\}_{n=1}^{\infty}$ converges (without worrying about what it actually converges *to* yet.) To do this, we need to just do two things: show that our sequence is **monotone** and that it is **bounded.**

Monotone-decreasing: we claim that

$$2^{\frac{1}{n+1}} < 2^{\frac{1}{n}}.$$

To see this, raise the left and right-hand-sides to the power $n(n+1)$, and simplify:

$$\begin{aligned} 2^{\frac{1}{n+1}} &< 2^{\frac{1}{n}} \\ \Leftrightarrow 2^{\frac{n(n+1)}{n+1}} &< 2^{\frac{n(n+1)}{n}} \\ \Leftrightarrow 2^{n} &< 2^{n+1} \\ \Leftrightarrow 1 &< 2. \end{aligned}$$

So our claim is equivalent to the inequality $1 < 2$, which is trivially true: so our sequence is monotonically decreasing, as claimed.

Bounded: Because our sequence is monotonically decreasing, it's bounded above by its first term, 2. So it suffices to find a lower bound.

We claim that 1 is a lower bound: i.e. that $2^{1/n} > 1$, for every n. This is trivially true: just raise both sides to the n-th power.

So our sequence is monotone and bounded: by the monotone-bounded convergence theorem, it must converge to some value L.

There are, in theory, three possibilities for L:

1. $L < 1$. In this case, we know that the values $2^{1/n}$ would have to converge to some value $L < 1$: in specific, we know that the $2^{1/n}$'s eventually are within ϵ of L, for any $\epsilon > 0$. So, take $\epsilon = 1 - L$: then, if there is a $2^{1/n}$ within this distance of L, we would have $2^{1/n} < 1$. But 1 is a lower bound: so this is impossible.

2. $L = 1$. This is what we claim is true: if it's the only possibility, we will have proven our claim.

3. $L > 1$. In this case, notice the following observation we can make about monotonically-decreasing sequences:

   Observation 2.5. If $\{a_n\}_{n=1}^{\infty}$ is a monotonically-decreasing sequence with limit L, then all of its terms are bounded below by L.

   This is because if there ever was a $a_N < L$, for some N, then every term after a_N will have to be at least that distance from L: i.e. $|a_n - L| > |a_N - L|$, for every $n > N$. This would force our sequence to always be at least $|a_n - L$ from L, and thus (in particular) not converge to L!

   What does this mean for our sequence? Well, it tells us that if $\lim_{n\to\infty} 2^{1/n} = L$, then L is a lower bound for the $2^{1/n}$'s. In specific, we have that

$$\begin{aligned} & 2^{1/n} > L \\ \Leftrightarrow \quad & 2 > L^{n}, \end{aligned}$$

for every $n \in \mathbb{N}$. But we've shown in class that if $L > 1$, $\lim_{n\to\infty} L^n = \infty$! So the L^n's cannot be bounded above by 2. Therefore, $L > 1$ is also impossible.

So we've proven that the only possible limit is $L = 1$, which was our claim! Thus, we've proven our result.

∎

Claim 4. (Squeeze theorem example:)

$$\lim_{n\to\infty} \frac{\sin\left(n^2 \cdot \pi^{n^e - 12n} \cdot n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}\right)}{n} = 0.$$

Proof. The idea of squeeze theorem examples is that they allow you to get rid of awful-looking things whenever they aren't materially changing where the sequence is actually going. Specifically, in our example here, the sin(terrible things) part is awful to work with, but really isn't doing anything to our sequence: the relevant part is the denominator, which is going to infinity (and therefore forcing our sequence to go to 0.

Rigorously: we have that

$$-1 \le \sin(\text{terrible things}) \le 1,$$

no matter what terrible things we've put into the sin function. Dividing the left and right by n, we have that

$$-\frac{1}{n} \le \frac{\sin(\text{terrible things})}{n} \le \frac{1}{n},$$

for every n. Then, because $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, the squeeze theorem tells us that

$$\lim_{n\to\infty} \frac{\sin\left(n^2 \cdot \pi^{n^e - 12n} \cdot n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}\right)}{n} = 0$$

as well. ∎

Claim 5. (Cauchy sequence example:) The sequence

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

converges.

Proof. To show that this sequence converges, we will use the Cauchy convergence tool, which tells us that sequences converge if and only if they are Cauchy.

How do we prove that a sequence is Cauchy? As it turns out, we can use a similar blueprint to the methods we used to show that a sequence converges:

- First, examine the quantity $|a_m - a_n|$, and try to come up with a very simple upper bound that depends on m and n and goes to zero. Example bounds we'd love to run into: $\frac{1}{n}, \frac{1}{mn}, \frac{1}{n}, \frac{1}{m^4 \log(n)}$. Things that won't work: $\frac{n}{m}$ (if n is really big compared to m, we're doomed!), $\frac{m}{n^{34}}$ (same!), 4.
- Using this upper bound, given $\epsilon > 0$, determine a value of N such that whenever $m > n > N$, our simple bound is less than ϵ.
- Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $m > n > N$, $|a_m - a_n| < \epsilon$.

Let's apply the above blueprint, and study $|a_m - a_n|$. Remember that we're assuming that $m > n$ here:

$$\begin{aligned} |a_m - a_n| &= \left| \sum_{k=1}^{m} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \right| \\ &= \sum_{k=n+1}^{m} \frac{1}{k^2} \end{aligned}$$

The following step may seem quite weird: it's motivated by partial fractions (because we want a way to simplify our $\frac{1}{k^2}$'s into simpler things), but it's mostly just an algebraic trick. The important thing is not to remember these tricks, but to just try tons of things until eventually *one* of them sticks:

$$\begin{aligned} \sum_{k=n+1}^{m} \frac{1}{k^2} &< \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\ &= \sum_{k=n+1}^{m} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ &= \sum_{k=n+1}^{m} \frac{1}{k-1} - \sum_{k=n+1}^{m} \frac{1}{k} \\ &= \sum_{k=n}^{m-1} \frac{1}{k} - \sum_{k=n+1}^{m} \frac{1}{k} \\ &= \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n}. \end{aligned}$$

This looks fairly simple!

Moving onto the second step: given $\epsilon > 0$, we want to force this quantity $\frac{1}{n} < \epsilon$. How can we do this? Well: if $m > n > N$, we have that $\frac{1}{n} < \frac{1}{N}$; so it suffices to pick N such that $\frac{1}{N} < \epsilon$.

Thus,we've shown that for any $\epsilon > 0$ we can find a N such that for any $m, n > N$,

$$|a_m - a_n| < \frac{1}{n} < \frac{1}{N} < \epsilon.$$

But this just means that our sequence is Cauchy! So, because all Cauchy sequences converge, we've proven that our sequence converges. $\square$

2.3 Series: Definitions and Tools

The last example we worked with (studying the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$) is actually our first example of a **series**. Specifically, we make the following definitions:

Definition 2.6. A sequence is called **summable** if the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums

$$s_n := a_1 + \dots a_n = \sum_{k=1}^{n} a_k$$

converges. If it does, we then call the limit of this sequence the **sum** of the a_n's, and denote this quantity by writing

$$\sum_{n=1}^{\infty} a_n.$$

We call such infinite sums **series**.

We say that a series$\sum_{n=1}^{\infty} a_n$ **converges** or **diverges** if the sequence $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ of partial sums converges or diverges, respectively.

Just like sequences, we have a collection of various tools we can use to study whether a given sequence converges or diverges:

1. **Comparison test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that $0 \leq a_n \leq b_n$, then the following statement is true:

   $$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

   When to use this test: when you're looking at something fairly complicated that either (1) you can bound above by something simple that converges, like $\sum 1/n^2$, or (2) that you can bound below by something simple that diverges, like $\sum 1/n$.

2. **Limit comparison test**: Suppose that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences of numbers such that $a_n \geq 0, b_n > 0$. Also, suppose that

   $$\lim_{n \to \infty} \frac{a_n}{b_n} = c \neq 0.$$

   Then the following statement is true:

   $$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Leftrightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

When to use this test: whenever you see something really complicated; so, mostly, in similar situations to the normal comparison test. The advantage to the limit comparison test is that you don't need your terms to always be bigger or smaller; so long as they look the same in the limit, you can use the limit comparison test. Really useful for reducing complicated polynomial expressions to their dominant terms.

3. **Alternating series test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers such that

   - $\lim_{n\to\infty} a_n = 0$ monotonically, and
   - the a_n's alternate in sign, then

   the series $\sum_{n=1}^{\infty} a_n$ converges. When to use this test: when you have an alternating series.

4. **Ratio test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

   $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

   then we have the following three possibilities:

   - If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
   - If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
   - If $r = 1$, then we have no idea; it could either converge or diverge.

   When to use this test: when you have something that is growing kind of like a geometric series: so when you have terms like 2^n or $n!$.

5. **Root test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

   $$\lim_{n\to\infty} \sqrt[n]{a_n} = r,$$

   then we have the following three possibilities:

   - If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
   - If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
   - If $r = 1$, then we have no idea; it could either converge or diverge.

   When to use this test: mostly, in similar situations to the ratio test. The ratio test is almost always easier: the only situation I can think of where the root test might be easier is when you have a n^n floating around without any other complicating terms like $n!$, and even then things like the comparison test will probably be easier, just because n^n grows so stupidly fast.

6. **Absolute convergence $\Rightarrow$ convergence**: Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence such that

   $$\sum_{n=1}^{\infty} |a_n|$$

converges. Then the sequence $\sum_{n=1}^{\infty} a_n$ also converges.

When to use this test: whenever you have a sequence that has positive and negative terms, that is not alternating. (Pretty much every other test requires that your sequence is positive, so you'll often apply this test and then apply one of the other tests to the series $\sum_{n=1}^{\infty} |a_n|$.)

2.4 Series: Applications of Convergence Tools

We'll have applications of the rest of these tools next week. For right now, we'll focus on an application of perhaps the simplest of our theorems (the alternating series test) that has perhaps the strangest results:

Claim 6. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges.

Proof. The terms in this series are alternating in sign: as well, they're bounded above and below by $\pm\frac{1}{n}$, both of which converge to 0. Therefore, we can apply the alternating series test to conclude that this series converges. $\square$

The reason we mention this sequence is because it illustrates some fantastically pathological behavior that series can exhibit. Specifically, think about what series **are**: just infinite sums of things! So: with finite sums, we know that addition has several nice properties. One particularly nice property that addition has is that it's **commutative**: i.e. the order in which we add things up doesn't matter!

A natural question we could ask, then, is the following: does this hold true with series? In other words, if we rearrange the terms in the series (say)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

will it still sum up to the same thing?

Well: let's try! Specifically, consider the following way to rearrange our series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots \\ &=^{?} \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) \dots\end{aligned}$$

In the second rearrangement, we've ordered terms in the following groups:

$$\dots + \left(\frac{1}{\text{odd number}} - \frac{1}{2 \cdot \text{that odd number}} - \frac{1}{2 \cdot \text{that odd number} + 2}\right) + \dots$$

Notice that every term from our original series shows up in exactly one of these groups. Specifically, each odd number clearly shows up once: as well, for any even number, there are two cases: either it has exactly one factor of 2, in which case it's of the form (2·an odd number) and shows up exactly once, or it's a multiple of 4, in which case it shows up as a (2·an odd number+2), and also shows up once. So this is in fact a proper rearrangement! We haven't forgotten any terms, nor have we repeated any terms.

But, if we group terms as indicated below in our rearrangement, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &=^? \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) \ldots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} \ldots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} \ldots \\
&= \frac{1}{2} \cdot \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \ldots\right) \\
&= \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}.
\end{aligned}$$

So: if rearranging terms doesn't change the sum of an infinte series, we've just shown that

$$\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}.$$

The only number that is equal to half of itself is 0: therefore, this series must sum to 0!

However: look at our series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ again. In specific, if we just expand this sum without rearranging anything, we can see that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \ldots \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \ldots \\
&= \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \frac{1}{9 \cdot 10} + \ldots \\
&\geq \frac{1}{2}.
\end{aligned}$$

So this sum definitely **cannot** converge to 0, as all of its partial sums are $\geq \frac{1}{2}$! This answers our question earlier about rearranging series fairly definitively: we've just shown that rearranging series can do **unpredictable, terrible, and horrible** things to the series itself.

Cool!