Math 8	Instructor: Padraic Bartlett

Lecture 10: Taylor Series

Week 10 — *Caltech - Fall, 2011*

1 Random Questions

Question 1.1. Show that

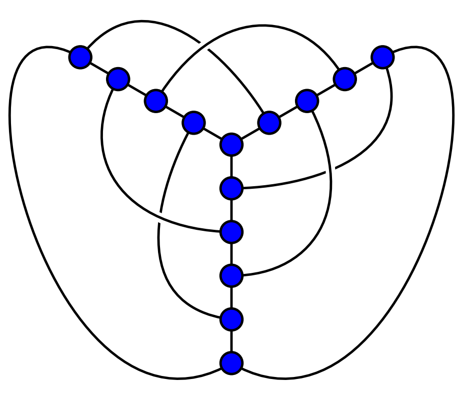
$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}} = \varphi, \text{ the golden ratio, which is } = \frac{1+\sqrt{5}}{2}.$$

Question 1.2. The complete bipartite graph $K_{n,n}$ is the following graph on $2n$ vertices: Consider the following graph:

- $V = X \cup Y, X = \{x_1, x_2, \ldots x_n\}, Y = \{y_1, y_2, \ldots y_n\}$.
- $E = \{\text{pairs of points } (x, y) \text{ such that } x \in X, y \in Y\}$.

Given a graph G, we can define a **drawing** of G as a way of associating the elements of V with distinct points in $\mathbb{R}^2$, and the elements (x, y) of E with curves drawn in the plane that connect the two points x and y.

Given a specific drawing of a graph, we can define the **crossing number** of this drawing as the number of places where two distinct edges cross each other. For example, in the drawing below, there are precisely 3 crossings:



The **crossing number of a graph** is the smallest crossing number of any drawing of that graph. Show that $K_{2,2}$ has crossing number 0, show that $K_{3,3}$ has crossing number at least 1, and try[1] to find the crossing number for $K_{n,n}$ in general.

[1]This question for $K_{n,n}$ is Turán's brick factory problem, a beautiful and currently open problem in mathematics.

2 Taylor Polynomials and Series

When we first introduced the idea of the derivative in week 5, one of the motivations we offered was the idea of the derivative $f'(x)$ as a sort of "linear approximation" to $f(x)$: essentially, given a function $f(x)$, the derivative $f'(x)$ was telling us the instantaneous slope of our function at the point x. Consequently, if we wanted to approximate our function at the point a, we could just use the function $f(a) + x \cdot f'(a)$; this "linear approximation" is often a decent replacement for the function itself, for values of x very close to a.

As mathematicians, we love overkill. So: if one derivative was useful, why not use n derivatives? This, roughly, is the motivation for **Taylor series and polynomials**:

Definition 2.1. Let $f(x)$ be a n-times differentiable function on some neighborhood $(a - \delta, a + \delta)$ of some point a. We define the **n-th Taylor polynomial of $f(x)$ around a** as the following degree-n polynomial:

$$T_n(f(x), a) := \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!} \cdot (x - a)^n.$$

Notice that this function's first n derivatives all agree with $f(x)$'s derivatives: i.e. for any $k \leq n$,

$$\frac{\partial^k}{\partial x^k} \left(T_n(f(x), a)\right)\Big|_a = f^{(k)}(a).$$

This motivates the idea of these Taylor polynomials as "n-th order approximations at a" of the function $f(x)$: if you only look at the first n derivatives of this function at a, they agree with this function completely.

We define the **n-th order remainder function of $f(x)$ around a** as the difference between $f(x)$ and its n-th order approximation $T_n(f(x), a)$:

$$R_n(f(x), a) = f(x) - T_n(f(x), a).$$

If f is an infinitely-differentiable function, and $\lim_{n\to\infty} R_n(f(x), a) = 0$ at some value of x, then we can say that these Taylor polynomials converge to $f(x)$, and in fact write $f(x)$ as its **Taylor series**:

$$T(f(x)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x - a)^n.$$

Often, we will assume that our Taylor series are being expanded around $a = 0$, and omit the a-part of the expressions above. If it is not specified, always assume that you're looking at a Taylor series expanded around 0.

One of the largest questions, given a function $f(x)$, is the following: at which values of x is $f(x)$ equal to its Taylor series? Equivalently, our question is the following: for what values of x is $\lim_{n\to\infty} R_n(f(x)) = 0$?

Our only/most useful tool for answering this question is the following theorem of Taylor:

Theorem 2.2. (Taylor's theorem:) If $f(x)$ is a $n+1$-times differentiable function on some neighborhood of a point a and $x > a$ is within this neighborhood, then

$$R_n(f(x), a) = \int_a^x \frac{f^{(n+1)}(t)}{n!} \cdot (x-t)^n dt.$$

In other words, we can express the remainder (a quantity we will often not understand) as an integral involving the derivatives of f divided by $n!$ (which is often easily bounded) and polynomials (which are also easy to deal with.)

The main use of Taylor series is the following observation, which basically states that integration and differentiation of Taylor series is amazingly easy:

Theorem 2.3. Suppose that $f(x)$ is a function with Taylor series

$$T(f(x)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n,$$

and furthermore suppose that $f(x) = T(f(x))$ on some interval $(-a, a)$. Then we can integrate and differentiate $f(x)$ by just termwise integrating and differentiating $T(f(x))$: i.e.

$$\frac{d}{dx} f(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \left(\frac{f^{(n)}(a)}{n!} \cdot (x-a)^n \right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{(n-1)!} \cdot (x-a)^{n-1}, \text{ and}$$
$$\int f(x)dx = \sum_{n=0}^{\infty} \int \left(\frac{f^{(n)}(a)}{n!} \cdot (x-a)^n \right) dx = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{(n+1)!} \cdot (x-a)^{n+1} + C.$$

In the following section, we study several functions, find their Taylor series, and use these results to perform calculations that are otherwise remarkably difficult:

3 Taylor Series: Applications

Proposition 3.1. The Taylor series for $f(x) = e^x$ about 0 is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Furthermore, this series converges and is equal to e^x on all of $\mathbb{R}$.

Proof. So: first, notice that

$$\frac{d^n}{dx}(e^x) = e^x,$$

and therefore that

$$T(e^x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{e^0}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Furthermore, by using Taylor's theorem, we know that the remainder term $R_n(e^x)$ is just

$$R_n(e^x) = \int_0^x \frac{f^{(n+1)}(t)}{n!} \cdot (x-t)^n dt = \int_0^x \frac{e^t}{n!} \cdot (x-t)^n dt.$$

Integrating this directly seems... painful. However, we don't need to know exactly what this integral is: we just need to know that it gets really small as n goes to infinity! So, instead of calculating this integral directly, we can just come up with some upper bounds on its magnitude.

Specifically: on the interval $[0, x]$, the function $|e^t|$ takes on its maximum at $t = x$, where it's e^x, and the function $|(x-t)^n|$ takes on its maximum at $t = 0$, where it's x^n. Therefore, we have that

$$\left| \int_0^x \frac{e^t}{n!} \cdot (x-t)^n dt \right| \leq \int_0^x \frac{e^x}{n!} \cdot x^n dt.$$

But the function being integrated at the right is just a constant with respect to t: there aren't any t's in it! This makes integration a lot easier:

$$\int_0^x \frac{e^x}{n!} \cdot x^n dt = \left(\frac{e^x}{n!} \cdot x^n \right) \cdot t \bigg|_0^x = e^x \cdot x \cdot \frac{x^{n+1}}{n!}.$$

Again: to show that e^x is equal to its Taylor series on all of $\mathbb{R}$, we just need to show that the remainder terms $R_n(e^x)$ always go to 0 as n goes to infinity. So, to finish our proof, it suffices to show that

$$\lim_{n\to\infty} e^x \cdot x \cdot \frac{x^n}{n!} = 0.$$

This is not hard to see. Specifically, pick any $k \geq 3 \in \mathbb{N}$, and let $n > kx$. Then we have that

$$\begin{aligned} \frac{x^n}{n!} &= \frac{x^{\lceil 2x \rceil}}{(\lceil 2x \rceil)!} \cdot \frac{x^{n-\lceil 2x \rceil}}{(\lceil 2x \rceil + 1) \cdot \ldots \cdot n} \\ &\leq x^{\lceil 2x \rceil} \cdot \frac{x \cdot x \cdot \ldots \cdot x}{2x \cdot 2x \cdot \ldots \cdot 2x} \\ &= x^{\lceil 2x \rceil} \cdot \frac{1}{2^{n-\lceil 2x \rceil}} \end{aligned}$$

Because $\lceil 2x \rceil$ is a fixed constant, letting n go to infinity in the above bound clearly goes to 0: therefore, we've proven that the remainder terms $R_n(e^x)$ go to 0 as n goes to infinity, and therefore that e^x is equal to its Taylor series everywhere. □

Using similar techniques, you can prove that the functions below have the following Taylor series, and furthermore converge to their Taylor series on the claimed sets:

Proposition 3.2.

$$T(\cos(x)) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ and } T(\cos(x)) = \cos(x) \text{ whenever } x \in \mathbb{R}.$$

$$T(\sin(x)) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ and } T(\sin(x)) = \sin(x) \text{ whenever } x \in \mathbb{R}.$$

$$T(\ln(1-x)) = \sum_{k=1}^{\infty} -\frac{x^n}{n}, \text{ and } T(\ln(1-x)) = \ln(1-x) \text{ whenever } x \in [-1, 1).$$

$$T\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} x^n, \text{ and } T\left(\frac{1}{1-x}\right) = \frac{1}{1-x} \text{ whenever } x \in (-1, 1).$$

$$\text{For } f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \ T\left(f(x)\right) = 0, \text{ and } T\left(f(x)\right) = f(x) \text{ only at } x = 0.$$

In addition, by substituting terms like $-x^2$ into the above Taylor series, we can derive Taylor series for other functions:

Proposition 3.3.

$$T(e^{-x^2}) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}, \text{ and } T(e^{-x^2}) = e^{-x^2} \text{ whenever } x \in \mathbb{R}.$$

$$T\left(\frac{1}{1+x^2}\right) = \sum_{k=0}^{\infty} (-1)^n x^{2n}, \text{ and } T\left(\frac{1}{1+x^2}\right) = \frac{1}{1+x^2} \text{ whenever } x \in (-1, 1).$$

Using Taylor series, we can approximate integrals that we could otherwise not calculate. For example, consider the Gaussian integral, which we ran into a few weeks ago on the HW:

$$\int e^{-x^2} dx.$$

Much to our frustration at the time, we saw that there was no "elementary" antiderivative for e^{-x^2}: in other words, there is no finite combination of functions like $\sin(x), e^x, x^n$ that will give an antiderivative of e^{-x^2}. This made working with this integral nigh-impossible: short of using rectangles and taking limits, we had no nice way of actually calculating any definite Gaussian integral!

Using Taylor series, however, we now **can** find this integral, to any level of precision we desire! We outline the method here, in the following example:

Question 3.4. Approximate

$$\int_0^2 e^{-x^2} dx$$

to within $\pm .1$ of its actual value.

Proof. Above, we proved that

$$T_n(e^x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Using this, we can write

$$e^{-x^2} = T_n(e^{-x})\Big|_{x^2} + R_n(e^{-x})\Big|_{x^2},$$

and therefore write

$$\int_0^2 e^{-x^2} dx = \int_0^2 T_n(e^{-x})\Big|_{x^2} dx + \int_0^2 R_n(e^{-x})\Big|_{x^2} dx.$$

Why is this nice? Well: the T_n part is just a polynomial: specifically, we have

$$T_n(e^{-x})\Big|_{x^2} = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{k!},$$

which is quite easy to integrate! As well, the R_n-thing is something that should be rather small for large values of n: so in theory we should be able to make its integral small, as well!

Specifically: using Taylor's theorem, we have that

$$R_n(e^{-x}) = \int_0^x \frac{e^{-t}}{n!} \cdot (x-t)^n dt$$

$$\Rightarrow R_n(e^{-x})\Big|_{x^2} = \int_0^{x^2} \frac{e^{-t}}{n!} \cdot (x^2-t)^n dt$$

Just like before, this is an integral we don't want to calculate: however, just like before, we don't have to! In particular, notice that on the interval $[0, x^2]$, the maximum value of e^{-t} is at $t = 0$, where it's $e^0 = 1$, and the maximum value of $(x^2 - t)^n$ is at $t = 0$, where it's x^{2n}. Therefore, we can bound the absolute value $\left|R_n(e^{-x})\Big|_{x^2}\right|$ of our remainder terms by

$$\int_0^{x^2} \frac{x^{2n}}{n!} dt = \left(\frac{x^{2n}}{n!}\right)\Bigg|_0^{x^2} = \frac{x^{2n+2}}{n!}.$$

Using this, we can finally bound the integral of our remainder terms:

$$\left|\int_0^2 R_n(e^{-x})\Big|_{x^2} dx\right| \leq \int_0^2 \frac{x^{2n+2}}{n!} dx = \frac{x^{2n+3}}{(2n+3)n!}\Bigg|_0^2 = \frac{2^{2n+3}}{(2n+3)n!}.$$

This quantity is $\leq .1$ at $n = 11$. Therefore, we've proven that

$$\int_0^2 e^{-x^2} dx = \int_0^2 T_{11}(e^{-x})\Big|_{x^2} dx,$$

up to $\pm .1$.

So: to find this integral, it suffices to integrate $T_n(e^{-x})\Big|_{x^2}$. This is pretty easy, if calculationally awkward:

$$\begin{aligned}
\int_0^2 T_{11}(e^{-x})\Big|_{x^2} dx &= \int_0^2 \sum^{1} 1_{k=0} \frac{(-1)^k x^{2k}}{k!} dx \\
&= \sum^{1} 1_{k=0} \int_0^2 \frac{(-1)^k x^{2k}}{k!} dx \\
&= \sum^{1} 1_{k=0} \left(\frac{(-1)^k x^{2k+1}}{(2k+1)k!} \right) \Bigg|_0^2 \\
&\approx .9\ .
\end{aligned}$$

So our integral is $.9 \pm .1$. □

Finally, to close our last regular lecture of the quarter, we perform a little bit of mathematical magic:

4 Magic

Above, we showed using Taylor series that we could express $\sin(x), \cos(x)$ and e^x as power series:

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots, \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots, \text{ and} \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots.
\end{aligned}$$

One thing you might have noticed when calculating these power series is that they look remarkably similar: specifically, the $\sin(x)$ power series looks like the odd parts of the e^x power series, while the $\cos(x)$ power series looks like the even parts of the e^x power series. Well, kinda: the $\sin(x)$ and $\cos(x)$ series alternate signs, while the e^x term does not.

Still. Is there any way we could somehow "fix" that alternating-sign thing, so that we could derive some sort of relation between $\sin(x), \cos(x)$, and e^x? In other words, is there anything we could plug into the $\sin(x), \cos(x)$ power series to make them not alternate sign?

Surprisingly, the answer is yes! Specifically, consider the complex number $i = \sqrt{-1}$! In particular, notice that

$$\{i^n\}_{n=0}^{\infty} = 1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \ldots,$$

and therefore that

$$\begin{aligned}
\sin(ix) &= ix - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \frac{(ix)^7}{7!} + \frac{(ix)^9}{9!} - \ldots \\
&= ix + i\frac{x^3}{3!} + i\frac{x^5}{5!} + i\frac{x^7}{7!} + i\frac{x^9}{9!} + \ldots, \text{ and} \\
\cos(ix) &= 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} - \frac{(ix)^6}{6!} + \frac{(ix)^8}{8!} - \ldots \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots.
\end{aligned}$$

Therefore, we have that $\cos(ix)$ is exactly the even terms of e^x's power series, while $\sin(ix)$ is just i times the odd terms of e^x's power series: i.e. that

$$e^x = \cos(ix) - i\sin(ix).$$

Plugging in $i\pi$ for x gives us in particular that

$$e^{i\pi} = \cos(-\pi) - i\sin(-\pi). = -1,$$

i.e.

$$e^{i\pi} + 1 = 0.$$

Which is **absolutely gorgeous.** In one succinct, beautiful formula, we've linked every single major mathematical constant together: $e, \pi, i, 1$, and 0. This equality is popularly known as Euler's formula: Richard Feynman once described it as "one of the most remarkable, almost astounding, formulas in all of mathematics."

It's a good place to finish our lectures for this year.