Math 8	Instructor: Padraic Bartlett

Review Session: Midterm

Week 5	*Caltech - Fall, 2011*

In Math 1, we've covered the following topics thus far throughout the course:

- **Proof methods**: we've discussed what it means to *prove* a mathematical statement, and discussed several techniques (contradiction, induction) that we can use to prove these kinds of claims.
- **$\mathbb{R}$ and $\mathbb{Q}$**: we've defined the rational numbers and real numbers, and discussed their properties, as fields and as ordered sets. We've also looked at the distinctions between rational and irrational numbers on several occasions.
- **Sequences, series, and power series**: we've defined sequences, series, and power series, defined a concept of **convergence** for all three objects, and developed tools for determining when these kinds of objects converge.
- **Continuity and limits**: we've defined the concepts of limits for real-valued functions, studied the concepts of continuity and discontinuity, and developed tools that help us tell when a function is continuous or discontinuous. As well, we've discussed a pair of applications of the concept of continuity: specifically, the intermediate value theorem and the extremal value theorem.

In these midterm review notes, we briefly go over each of these topics, listing key definitions, theorems, and working an example for each topic. We start with proof methods:

1 Proof Methods

(Relevant lectures: Week 1's discussion of proof methods.)

The "idea" of proof is something you all understand quite well by now: the only thing that it's probably useful to review is how to do proofs by induction. We do this below:

1.1 Proofs by Induction

Suppose that you have a claim $P(n)$ – a sentence like "$2^n \geq n$", for example. How do we prove that this kind of thing holds by induction? Essentially, we just need to follow the outline below:

Claim 1. $P(n)$ holds, for all $n \geq k$.

Proof.
Base case: we prove (by hand) that $P(k)$ holds, for a few base cases.
Inductive step: Assuming that $P(n)$ holds, we prove that $P(n+1)$ holds. Often, we do this by breaking our $P(n+1)$ claim apart into something that looks like $P(n)$, along with some extra bits.

1.2 Examples

Conclusion: $P(n)$ holds for all $n \geq k$. $\square$

Example. Suppose that $a_1, \ldots a_n$ are all real numbers from the interval $[0,1]$. Then

$$\prod_{i=1}^{n}(1-a_i) \geq 1 - \sum_{i=1}^{n} a_i.$$

Proof. We proceed by induction. Our base case is trivial: for $n = 1$, our claim is

$$\prod_{i=1}^{1}(1-a_i) = 1 - a_1 \geq 1 - a_1 = 1 - \sum_{i=1}^{1} a_i,$$

which is trivially true.

Inductive step: assume that our claim holds for n. We seek to prove our claim for $n+1$: i.e. take any collection of $n+1$ values $a_1, \ldots a_{n+1}$. Examine the product

$$\prod_{i=1}^{n+1}(1-a_i) = (1-a_{n+1}) \cdot \left(\prod_{i=1}^{n}(1-a_i)\right).$$

If we apply our inductive hypothesis to $(\prod_{i=1}^{n}(1-a_i))$, we can see that

$$\begin{aligned}
\prod_{i=1}^{n+1}(1-a_i) &= (1-a_{n+1}) \cdot \left(\prod_{i=1}^{n}(1-a_i)\right) \\
&\geq (1-a_{n+1}) \cdot \left(1 - \sum_{i=1}^{n} a_i\right) \\
&= 1 - a_{n+1} - \left(\sum_{i=1}^{n} a_i\right) + a_{n+1} \cdot \left(\sum_{i=1}^{n} a_i\right) \\
&\geq 1 - a_{n+1} - \left(\sum_{i=1}^{n} a_i\right) \\
&= 1 - \sum_{i=1}^{n+1} a_i.
\end{aligned}$$

Notice that we used that all of the a_i's were ≤ 1 when we plugged in our inductive hypothesis (as otherwise $(1-a_{n+1})$ would have been negative, and our inequality would have gotten reversed.) Also notice that we used that all of the $a_i's$ were positive when we said that $a_{n+1} \cdot (\sum_{i=1}^{n} a_i) \geq 0$ and therefore getting rid of it only makes the right-hand-side smaller. $\square$

2 The Real and Rational Number Systems

(Relevant lectures: Week 2's discussion of the real and rational number systems.)

2.1 Definitions / Basic Results

We've studied a lot of things about the real and rational number systems. Most of these aren't going to show up on the midterm; however, you should be aware of the following definitions and concepts:

Definition. The **real numbers** can be defined as the collection of all strings of the form

$$x_{int} \,.\, x_1 x_2 x_3 \ldots,$$

wherex_{int} is an integer and the values x_i are all decimal digits: i.e. elements of the set $\{0, 1, 2, \ldots 9\}$. In simpler terms, the real numbers are the collection of all infinite decimal expansions.

We have several well-defined binary operations on the real numbers: addition, multiplication, subtraction, and division (defined whenever the denominator is nonzero.) We also have a well-defined concept of **order** for the real numbers, which allows us to compare any two distinct real numbers and determine which is larger. These operations all satisfy several properties, called the **field axioms** and **order axioms**: I would not recommend attempting to memorize them or even really looking at them very much, as they're somewhat dull, but I would recommend having the page with their definitions bookmarked in Apostol, or something like that.

The **rational numbers** are the subset of the real numbers defined as the following set:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, GCD(a, b) = 0, b > 0 \right\}.$$

The rational numbers satisfy many of the same properties as the real numbers: in specific, with respect to the operations $+, \cdot, -, /, <$, the rational numbers satisfy the same field axioms and order axioms as the real numbers.

Both the real and rational numbers satisfy the following property:

Proposition 2.1. *(Archimedean property, version 1:) For any $x > 0, y > 0$, there is some $n \in \mathbb{N}$ such that $nx > y$.*

We proved in Math 8 that this statement was equivalent to the following:

Proposition 2.2. *(Archimedean property, version 2:) For any $x > 0$, there is some $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.*

We have proven that there are numbers that are real but not rational:

Theorem 2. The constants $\sqrt{2}, \sqrt[3]{2}, \sqrt{3} - \sqrt{5}, e$ are all irrational.

We have also proven that the irrational and rational numbers are **dense** in the real number line:

Theorem 3. For any pair of distinct real numbers $x < y$, there is an irrational number a and rational number b such that $x < a < y$ and $x < b < y$.

For the most part, questions that ask you to work with the real and rational number systems will feel like the proofs we did to show that certain numbers were irrational: they will work heavily off of the **definitions** of these two sets, and not use the results we've proven about these two sets as much. The following example should illustrate the kind of thing we might ask:

2.2 Examples

Example. Show that a number $x \in \mathbb{R}$'s decimal expansion is repeating if and only if it is rational.

Proof. We say that a real number x has a **repeating decimal expansion** if and only if there is some power of ten 10^k such that we can write $x \cdot 10^k$ in the following form:

$$x \cdot 10^k = x_{int} \,.\, x_1x_2 \ldots x_nx_1x_2 \ldots x_nx_1x_2 \ldots x_n \ldots$$

(The reason for the 10^k is that we want to think of numbers like $.063333333\ldots$ as repeating, even though their first two decimal places aren't part of the pattern.)

So: How do we go about this proof?

One direction is not terribly difficult: suppose that x is a real number of the form above. Then, we have that

$$\begin{aligned}
x \cdot 10^k \cdot 10^n &= x_{int}x_1x_2 \ldots x_n \,.\, x_1x_2 \ldots x_nx_1x_2 \ldots x_n \ldots \\
\Leftrightarrow x \cdot 10^k \cdot 10^n - x \cdot 10^k &= (x_{int}x_1x_2 \ldots x_n \,.\, x_1x_2 \ldots x_nx_1x_2 \ldots x_n \ldots) - (x_{int} \,.\, x_1x_2 \ldots x_nx_1x_2 \ldots x_n) \\
&= \text{(an integer)} \\
\Leftrightarrow x \cdot \left(10^{n+k} - 10^k\right) &= \text{(an integer)} \\
\Leftrightarrow x &= \frac{\text{(an integer)}}{10^{n+k} - 10^k}.
\end{aligned}$$

So, being a number with a repeating decimal expansion is equivalent to being a number of the form $\frac{\text{(an integer)}}{10^{n+k}-10^k}$. This tells us that all numbers with repeating decimal expansions are rational, as we've expressed them as a ratio of two integers.

The other direction is not much trickier. Suppose that $x = \frac{p}{q}$ for some pair of relatively prime integers p, q, with $q > 0$. Now, look at powers of 10 mod q: in other words, take the collection of numbers

$$1, 10, 100, 1000, \ldots$$

and look at their remainders after you divide them by q. Because there are infinitely many powers of 10 and only finitely many numbers that can be remainders when we divide by q (specifically, there are only the numbers $\{0 \ldots q-1\}$,) we know that there are two natural numbers $n > m$ such that 10^n and 10^m have the same remainder when divided by q: in other words,

$$10^n - 10^m = q \cdot z,$$

for some integer z.

But this means that

$$\left(\frac{p}{q}\right) \cdot (10^n - 10^m) = p \cdot z \in \mathbb{Z} :$$

i.e. that $x \cdot 10^n - x \cdot 10^m$ is an integer, and thus that

$$x = \frac{\text{(an integer)}}{10^n - 10^m}.$$

But we just showed that this was equivalent to having a repeating decimal expansion above! So we've shown that all rational numbers have repeating decimal expansions. $\square$

As a corollary, we've proven the following:

Corollary 2.3. *The number*

$$.0100100010000100000100000001\ldots$$

is irrational, because its decimal expansion is nonrepeating by design.

3 Sequences / Series / Power Series

(Relevant lectures: Week 3's discussion of sequences and series, week 4's discussion of power series.)

3.1 Definitions / Basic Results

A **sequence** is just an infinite collection of objects $\{a_n\}_{n=1}^{\infty}$ indexed by the natural numbers. The main property that we've studied about sequences in this class is that of convergence:

Definition 3.1. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value λ if, for any distance ϵ, the a_n's are eventually within ϵ of λ. To put it more formally, $\lim_{n\to\infty} a_n = \lambda$ iff for any distance ϵ, there is some cutoff point N such that for any n greater than this cutoff point, a_n must be within ϵ of our limit λ.

In symbols:

$$\lim_{n\to\infty} a_n = \lambda \text{ iff } (\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

We have the following tools for determining when a sequence converges:

1. **The definition of convergence**: The simplest way to show that a sequence converges is sometimes just to use the definition of convergence. In other words, you want to show that for any distance ϵ, you can eventually force the a_n's to be within ϵ of our limit, for n sufficiently large.

2. **Arithmetic and sequences**:

   - *Additivity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$.
   - *Multiplicativity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
   - *Quotients of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$.

3. **Monotone and bounded sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded below and nonincreasing, it also converges.

4. **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l.

5. **Cauchy sequences**: We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m > n \geq N$, we have

$$|a_m - a_n| < \epsilon.$$

The Cauchy theorem, in this situation, is the following: a sequence is Cauchy if and only if it converges.

Definition 3.2. A sequence is called **summable** if the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums

$$s_n := a_1 + \ldots a_n = \sum_{k=1}^{n} a_k$$

converges. If it does, we then call the limit of this sequence the **sum** of the a_n's, and denote this quantity by writing

$$\sum_{n=1}^{\infty} a_n.$$

We call such infinite sums **series**.

We say that a series$\sum_{n=1}^{\infty} a_n$ **converges** or **diverges** if the sequence $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ of partial sums converges or diverges, respectively.

Just like sequences, we have a collection of various tools we can use to study whether a given sequence converges or diverges:

1. **Comparison test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that $0 \leq a_n \leq b_n$, then the following statement is true:

$$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

2. **Limit comparison test**: Suppose that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences of numbers such that $a_n \geq 0, b_n > 0$. Also, suppose that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0.$$

Then the following statement is true:

$$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Leftrightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

3. **Alternating series test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers such that
   - $\lim_{n\to\infty} a_n = 0$ monotonically, and
   - the a_n's alternate in sign, then

   the series $\sum_{n=1}^{\infty} a_n$ converges.

4. **Ratio test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

   then we have the following three possibilities:
   - If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
   - If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
   - If $r = 1$, then we have no idea; it could either converge or diverge.

5. **Root test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = r,$$

   then we have the following three possibilities:
   - If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
   - If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
   - If $r = 1$, then we have no idea; it could either converge or diverge.

6. **Absolute convergence $\Rightarrow$ convergence**: Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence such that

$$\sum_{n=1}^{\infty} |a_n|$$

   converges. Then the sequence $\sum_{n=1}^{\infty} a_n$ also converges.

Definition 3.3. A **power series** $P(x)$ centered at x_0 is just a sequence $\{a_n\}_{n=1}^{\infty}$ written in the following form:

$$P(x) = \sum_{n=0}^{\infty} a_n \cdot (x - x_0)^n.$$

Power series are almost taken around $x_0 = 0$: if x_0 is not mentioned, feel free to assume that it is 0.

A power series $P(x) = \sum_{n=1}^{\infty} a_n x^n$ is said to **converge** at a value b if the series gained by plugging b into $P(x)$, i.e. the series $\sum_{n=1}^{\infty} a_n b^n$, converges.

We have the following two results about power series and convergence:

Theorem 4. Suppose that

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series that converges at some value $b \in \mathbb{R}$. Then $P(x)$ actually converges on every value in the interval $(-b, b)$.

Corollary 5. Suppose that

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a power series centered at 0, and A is the set of all real numbers on which $P(x)$ converges. Then there are only three cases for A: either

1. $A = \{0\}$,
2. $A =$ one of the four intervals $(-b, b), [-b, b), (-b, b], [-b, b]$, for some $b \in \mathbb{R}$, or
3. $A = \mathbb{R}$.

We say that a power series $P(x)$ has **radius of convergence** 0 in the first case, b in the second case, and ∞ in the third case.

To illustrate these ideas, we work some examples:

3.2 Examples

Example. Show that the sequence

$$1, \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$$

converges.

Proof. Another way of describing this sequence is as the sequence whose first term a_1 is 1, and which satisfies the following property:

$$a_{n+1} = \sqrt{2 \cdot a_n}.$$

We make two claims about this sequence:

1. It is bounded above by 2. The proof is a simple induction: for a_1, this is immediate. For our inductive step, we assume that $a_n < 2$, and examine a_{n+1}: using our inductive hypothesis, we have

   $$a_{n+1} = \sqrt{2 \cdot a_n} < \sqrt{2 \cdot 2} = 2.$$

   So we've proven that our sequence is bounded above by 2.

2. It is monotone-increasing: i.e. $a_n < a_{n+1}$, for every n. This is another induction. Our base case ($a_1 < a_2$) is trivial; if we assume our inductive claim ($a_{n-1} < a_n$) and look at the terms a_n, a_{n+1}, we can use our inductive hypothesis to see that

$$\begin{aligned} a_n^2 &= \left(\sqrt{2a_{n-1}}\right)^2 = 2a_{n-1} < 2a_n = \left(\sqrt{2a_n}\right)^2 = a_{n+1}^2 \\ \Rightarrow a_n &< a_{n+1}. \end{aligned}$$

So our sequence must converge, as it is monotone-increasing and bounded above. □

Example. Let $\alpha(n)$ denote the function that takes in any natural number n and returns the number of prime factors it has, counting repeated primes. For example, $\alpha(6) = 2, \alpha(8) = 3, \alpha(120) = 5, \alpha(17) = 1$. Show that the series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^3}$$

converges.

Proof. To do this, simply notice that $\alpha(n) < n$ for any n, as the number of prime factors in a number can never exceed that number. (Why? Well, suppose that you have a number with k prime factors. Then this number is at least 2^k, because each prime factor contributes a multiplier of at least 2, the smallest prime. We know that $k < 2^k$, for all k, so this proves our claim.)

Using this, notice that we have

$$\frac{\alpha(n)}{n^3} < \frac{n}{n^3} = \frac{1}{n^2},$$

for all $n \in \mathbb{N}$: applying the comparison theorem then tells us that because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so must our series

$$\sum_{n=1}^{\infty} \frac{\alpha n}{n^3}.$$

□

Example. Find the radius of convergence of the power series

$$P(x) = \sum_{n=1}^{\infty} \frac{\alpha(n) \cdot x^n}{n^3},$$

where $\alpha(n)$ is defined as above.

Proof. First, we apply the comparison test (because the $\alpha(n)$ bit is irritating) and compare $P(x)$ to the series $\sum^{\infty} \frac{x^n}{n^2}$, which we can do because each of these terms are strictly greater than the terms in $P(x)$. The comparison test then tells us that $P(x)$ will converge whenever

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

converges. To determine when this happens, we apply the ratio test, and look at the limit

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)^2}{x^n/n^2} = \lim_{n\to\infty} \cdot \left(\frac{n}{n+1}\right)^2 \cdot x = \lim_{n\to\infty} \cdot \left(1 - \frac{1}{n+1}\right)^2 \cdot x = x.$$

The ratio test says that this power series converges for all positive $x < 1$, which means our original series $P(x)$ converges for all positive $x < 1$. Because this is a power series, we know that this means it must converge on all of $(-1, 1)$.

We claim that $P(x)$ has radius of convergence 1: i.e. that it will diverge if we put in any value of $x > 1$. To see why, apply the comparison theorem **again**, but this time compare $P(x)$ to the series $\sum^{\infty} \frac{x^n}{n^3}$, all of whose terms are strictly smaller than the terms in $P(x)$. The comparison test now tells us that $P(x)$ will **diverge** whenever

$$\sum_{n=1}^{\infty} \frac{x^n}{n^3}$$

diverges. To see where that is, simply apply the ratio test again:

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)^3}{x^n/n^3} = \lim_{n\to\infty} \cdot \left(\frac{n}{n+1}\right)^3 \cdot x = \lim_{n\to\infty} \cdot \left(1 - \frac{1}{n+1}\right)^3 \cdot x = x,$$

which is greater than 1 whenever $x > 1$. So we have that this series diverges whenever $x > 1$, and therefore that $P(x)$ diverges for $|x| > 1$ and converges for $|x| < 1$: i.e. its radius of convergence is 1. $\square$

4 Limits and Continuity

(Relevant lectures: Week 4's lectures on continuity, week 5's discussion of a particular discontinuous function.)

4.1 Definitions

In the fourth week of our course, we turned to the study of limits of functions; here, we encountered our first $\epsilon - \delta$ proofs, and began to work with the notion of continuity. We review several key definitions here:

Definition 4.1. If $f : X \to Y$ is a function between two subsets X, Y of $\mathbb{R}$, we say that

$$\lim_{x\to a} f(x) = L$$

if and only if

1. (vague:) as x approaches a, $f(x)$ approaches L.

2. (precise; wordy:) for any distance $\epsilon > 0$, there is some neighborhood $\delta > 0$ of a such that whenever $x \in X$ is within δ of a, $f(x)$ is within ϵ of L.

3. (precise; symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x-a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

We have several generalizations of this idea, which we list here:

1. **Continuous**: A function $f : X \to Y$ is said to be **continuous** at some point $a \in X$ iff

$$\lim_{x \to a} f(x) = f(a).$$

2. **One-sided limits**: we say that $\lim_{x\to a^+} f(x) = L$ if and only if "when x goes to a from the right-hand-side, $f(x)$ goes to L:" i.e.

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x-a| < \delta \text{ and } x > a) \Rightarrow (|f(x) - L| < \epsilon).$$

The left-hand-side limit $\lim_{x\to a^-} f(x) = L$ is defined similarly.

3. **Limits at infinity**: for a function $f : X \to Y$, we say that $\lim_{x\to+\infty} f(x) = L$ if and only if "when x goes to "infinity," $f(x)$ goes to L: i.e.

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x > N) \Rightarrow (|f(x) - L| < \epsilon).$$

As before, we developed several useful tools and blueprints for dealing with limits, which we review here:

1. **A blueprint for $\epsilon - \delta$ proofs of limits**: In class, we developed the following "blueprint" that describes a general method for proving that $\lim_{x\to a} f(x) = L$ via an $\epsilon - \delta$ argument. We review this below:

   (a) First, examine the quantity

$$|f(x) - L|.$$

   Specifically, try to find a simple upper bound for this quantity that depends only on $|x - a|$, and goes to 0 as x goes to a – something like $|x-a| \cdot$ (constants), or $|x - a|^3 \cdot$ (bounded functions, like $\sin(x)$).

   (b) Using this simple upper bound, for any $\epsilon > 0$, choose a value of δ such that whenever $|x-a| < \delta$, your simple upper bound $|x-a| \cdot$ (constants) is $< \epsilon$. Often, you'll define δ to be $\epsilon/$(constants), or somesuch thing.

   (c) Plug in the definition of the limit: for any $\epsilon > 0$, we've found a δ such that whenever $|x - a| < \delta$, we have

$$|f(x) - L| < (\text{simple upper bound depending on } |x-a|) < \epsilon.$$

   Thus, we've proven that $\lim_{x\to a} f(x) = L$, as claimed.

2. **A blueprint for proving that certain limits do *not* exist**: In class, we proved the following lemma:

Lemma 4.2. *For any function $f : X \to Y$, we know that $\lim_{x \to a} f(x) \neq L$ iff there is some sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:*

- $\lim_{n \to \infty} a_n = L$, *and*
- $\lim_{n \to \infty} f(a_n) \neq L$, *and*

This lemma makes proving that a function f is discontinuous at some point a remarkably easy:

- to prove that $\lim_{x \to a} f(x) \neq L$,
- all we have to do is just find *one* sequence $\{a_n\}_{n=1}^{\infty}$ that converges to a, such that $\lim_{n \to \infty} f(a_n) \neq L$ *on that sequence*! Basically, it allows us to work in the world of sequences instead of that of continuity, which can make life a lot easier on us.

3. **Squeeze theorem**: If f, g, h are functions defined on some interval $I \setminus \{a\}$[1] such that

$$\begin{aligned} &f(x) \leq g(x) \leq h(x), \forall x \in I \setminus \{a\}, \text{ and} \\ &\lim_{x \to a} f(x) = \lim_{x \to a} h(x), \end{aligned}$$

then $\lim_{x \to a} g(x)$ exists, and is equal to the other two limits $\lim_{x \to a} f(x), \lim_{x \to a} h(x)$.

4. **Limits and arithmetic**: If f, g are a pair of functions such that $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ both exist, then we have the following equalities:

$$\begin{aligned} \lim_{x \to a} (\alpha f(x) + \beta g(x)) &= \alpha \left(\lim_{x \to a} f(x) \right) + \beta \left(\lim_{x \to a} g(x) \right) \\ \lim_{x \to a} (f(x) \cdot g(x)) &= \left(\lim_{x \to a} f(x) \right) \cdot \left(\lim_{x \to a} g(x) \right) \\ \lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) &= \left(\lim_{x \to a} f(x) \right) / \left(\lim_{x \to a} g(x) \right), \text{ if } \lim_{x \to a} g(x) \neq 0. \end{aligned}$$

5. **Limits and composition**: If $f : Y \to Z$ is a function such that $\lim_{y \to a} f(x) = L$, and $g : X \to Y$ is a function such that $\lim_{x \to b} g(x) = a$, then

$$\lim_{x \to b} f(g(x)) = L.$$

[1] The set $X \setminus Y$ is simply the set formed by taking all of the elements in X that are not elements in Y. The symbol $\setminus$, in this context, is called "set-minus", and denotes the idea of "taking away" one set from another.

4.2 Examples

Example. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} \frac{p-q}{pq}, & x \in \mathbb{Q}, x = \frac{p}{q}, GCD(p,q) = 1, q > 0. \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Show that $f(x)$ is continuous at 1, continuous at π, and discontinuous at 2.

Proof. First, notice that if we're expressing 1,2 as fractions with relatively prime numerators and denominators, we have $1 = \frac{1}{1}, 2 = \frac{2}{1}$, and therefore

$$f(1) = \frac{1-1}{1 \cdot 1} = 0, \quad f(2) = \frac{2-1}{2 \cdot 1} = \frac{1}{2}.$$

To see that our function is discontinuous at 2: simply notice that there is a sequence $x_n = 2 + \frac{\pi}{10^n}$, such that

- $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 2 + \frac{\pi}{10^n} = 2$, and
- $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(\text{irrationals}) = \lim_{n\to\infty} 0 = 0 \neq f(2) = \frac{1}{2}$.

Therefore, by our lemma about discontinuity, our function cannot be continuous at 2.

To see that our function is continuous at 1: pick any $\epsilon > 0$. We want to find a $\delta > 0$ such that whenever $|x - 1| < \delta$, we have $|f(x) - 0| < \epsilon$.

How can we do this? Well: notice that because $f(\text{irrationals}) = 0$, we never have to worry about the irrational points: whenever $x \notin \mathbb{Q}$, we have $|f(x) - 0| = 0 < \epsilon$.

So we only need to worry about rational points. Pick n such that $\epsilon > \frac{1}{n}$. What can we say about any rational point within $\frac{1}{n}$ of 1, that's not equal to 1? Well: we know that both its numerator and denominator must be greater than n, as otherwise (because it's not equal to 1) we'd have that it was further than $\frac{1}{n}$ from 1. This then tells us that for any such rational $x = \frac{p}{q}, |x - 1| < \frac{1}{n}$, we have

$$|f(x)| = \left|\frac{p-q}{pq}\right| = \left|\frac{1}{q} - \frac{1}{p}\right| < \left|\frac{1}{n} - 0\right| < \epsilon.$$

So, if we pick $\delta = \frac{1}{n} < \epsilon$, we've proven our claim: for any x within δ of 1, we have shown that $|f(x) - 1| < \epsilon$. So our function is continuous at 1.

To see that our function is also continuous at π, proceed similarly: pick any $\epsilon > 0$. We want to find a δ such that whenever $|x - \pi| < \delta$, we have $|f(x) - f(\pi)| = |f(x)| < \epsilon$. Again, as before, we can ignore the irrational points, and only need to worry about the rational points.

At rational points, we know that

$$|f(x)| = \left|\frac{p-q}{pq}\right| = \left|\frac{1}{q} - \frac{1}{p}\right|.$$

What kinds of rational points make this quantity small? Well: as noticed before, if we can force our rational points to all have sufficiently large numerators and denominators, then this quantity will be as small as we like!

We make this rigorous here. For any N, notice that there are finitely many rational numbers of the form $\frac{p}{q}$, where one of p, q is $< N$, and $\frac{p}{q}$ lies in the interval $[\pi - 1, \pi + 1]$. Let A be the set of all of these rational numbers. Because it is finite, there is some rational number $\frac{p'}{q'}$ in A that is **closest** to π: let δ be the distance from $\frac{p'}{q'}$ to π. Then, for any $\frac{p}{q}$ within δ of π, we have

$$|f(x)| = \left|\frac{p-q}{pq}\right| = \left|\frac{1}{q} - \frac{1}{p}\right| < \frac{1}{N} + \frac{1}{N} = \frac{2}{N}.$$

So: pick N large enough such that $\epsilon > \frac{2}{N}$. Then, if we choose δ as described above, we will have forced $|f(x)| < \epsilon$, which is the goal of our epsilon-delta proof. $\square$