MATH 8, SECTION 1, WEEK 9 - RECITATION NOTES

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Abstract. These are the notes from Wednesday, Nov. 23rd's lecture, where we continued our study of Taylor series.

1. Random Question

Question 1.1. *Can you arrange the sixteen cards* $\{A, K, Q, J\} \times \{\heartsuit, \spadesuit, \clubsuit, \diamondsuit\}$ *into a* 4×4 *grid, so that none of the suits or symbols are repeated in any row or column?*

How about for general n *- i.e. if you have* n *symbols* $\{\square_1, \dots \square_n\}$ *and* n *suits* $\{\triangle_1, \dots \triangle_n\}$*, can you arrange the* n^2 *cards* $\{\square_1, \dots \square_n\} \times \{\triangle_1, \dots \triangle_n\}$ *into a* $n \times n$ *grid such that no symbol nor suit is repeated in any of those rows?*

(Hint: General n *is rather tricky. Try considering some special cases:* 2×2*,* 5×5*,* 6×6*,* 10×10*.)*

2. Taylor Series: Composition and Applications

We have discovered the Taylor polynomials for several functions in class. For convenience's sake, we relist them here:

$$
\begin{aligned}
T_{2n}(\cos(x), 0) &= \sum_{k=0}^{n} (-1)^n \frac{x^{2n}}{(2n)!}, \\
T_{2n+1}(\sin(x), 0) &= \sum_{k=0}^{n} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \\
T_n(e^x, 0) &= \sum_{k=0}^{n} \frac{x^k}{k!} \\
T_n(\log(x+1), 0) &= \sum_{k=1}^{n} (-1)^{n+1} \frac{x^n}{x} \\
T_n((x+y)^a, 0) &= \sum_{k=0}^{n} \binom{a}{k} x^k y^{a-k}
\end{aligned}
$$

Can we use these "known" Taylor polynomials to derive Taylor polynomials for other functions? For example: we know $T_n(\cos(x), 0)$. Can we use this to derive $T_n(\cos(ax^m, 0)$, for any $a, m \in \mathbb{R}$?

As it turns out, yes! We in fact have the following:

Lemma 2.1.

$$T_{2n\cdot m}(\cos(ax^m), 0) = \sum_{k=0}^{n} (-1)^n \frac{(ax^m)^{2n}}{(2n)!}.$$

In other words, we can get the Taylor polynomial of $\cos(ax^m)$ *by simply composing* ax^m *with* $\cos(x)$.

Proof. To show this, we merely need to show that $\cos(ax^m)$ and $\sum_{k=0}^{n}(-1)^n \frac{(ax^m)^{2n}}{(2n)!}$ agree up to order $2mn$ at 0 – i.e. that

$$\lim_{x\to 0} \frac{\cos(ax^m) - \sum_{k=0}^{n}(-1)^n \frac{(ax^m)^{2n}}{(2n)!}}{x^{2mn}} = 0.$$

But this is immediate, as we can see by letting $y = ax^m$:

$$\begin{aligned}
&\lim_{x\to 0} \frac{\cos(ax^m) - \sum_{k=0}^{n}(-1)^n \frac{(ax^m)^{2n}}{(2n)!}}{x^{2mn}} \\
=&\lim_{y\to 0} \frac{\cos(y) - \sum_{k=0}^{n}(-1)^n \frac{(y)^{2n}}{(2n)!}}{y^{2n}/a^{2n}} \\
=&a^{2n} \cdot \lim_{y\to 0} \frac{\cos(y) - \sum_{k=0}^{n}(-1)^n \frac{(y)^{2n}}{(2n)!}}{y^{2n}},
\end{aligned}$$

and we know that the limit on the inside is 0 because the $2n$-th Taylor polynomial for $\cos(y)$ is in fact $\sum_{k=0}^{n}(-1)^n \frac{(y)^{2n}}{(2n)!}$. As a consequence, we have that these two functions agree up to order x^{2mn}, and thus that

$$T_{2n\cdot m}(\cos(ax^m), 0) = \sum_{k=0}^{n} (-1)^n \frac{(ax^m)^{2n}}{(2n)!},$$

as claimed. ☐

So: we can compose functions with Taylor series! How can we use this composition process?

2.1. **A Completely Useless Taylor Series.** In class, someone asked if there was a function whose Taylor series *never* converged: i.e. if there was a function $f(x)$, infinitely differentiable, such that

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k$$

diverged for every value of x. As it turns out, this is impossible: if $x = a$, we have that

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} \cdot (a-a)^k = f(a),$$

and thus that this series converges.

Well, that's a somewhat silly objection. Here's a potentially more interesting question: is there a function whose Taylor series around a diverges for any value of $x \neq a$?

As it turns out, yes!

Lemma 2.2. *The Taylor series around 0 of the function*

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(2^k x)}{k!}$$

diverges for any value of $x \neq 0$.

Proof. As it stands, we unfortunately don't have enough mathematical firepower to prove this directly (at least, to prove this directly within a single lecture!) So: in order to prove this, we will use without proof the following fact:

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}\left(\sum_{n=0}^{\infty} \frac{\cos(2^k x)}{k!}\right) &= \sum_{n=0}^{\infty} \frac{\partial^k}{\partial x^k}\left(\frac{\cos(2^k x)}{k!}\right) \\
&= \sum_{n=0}^{\infty} \frac{\partial^k}{\partial x^k}\left(\frac{\sum_{j=0}^{\infty}(-1)^j \frac{(2^k x)^{2j}}{(2j)!}}{k!}\right) \\
&= \sum_{n=0}^{\infty} \frac{\sum_{j=0}^{\infty}(-1)^j \cdot \frac{\partial^k}{\partial x^k}\left(\frac{(2^k x)^{2j}}{(2j)!}\right)}{k!}
\end{aligned}$$

(Justifying this requires using things like "uniform convergence," a concept we haven't discussed yet but which is not beyond your current abilities! See the notes from Ma1d if you're curious to see how this plays out.)

Taking this observation – that, in certain cases, derivatives can pass through infinte sums – for granted, we can easily calculate the derivatives of f at 0.

First: recall that the n-th derivative of x^j, evaluated at 0, is equal to 0 if $n \neq j$, and $n!$ if $n = j$. This is because

- if $n > j$, then taking n derivatives of x^j will reduce it to 0,
- if $n < j$ then taking n derivatives will leave $j \cdot \ldots \cdot (j - n + 1) \cdot x^{n-j}$, which is 0 at 0, and
- if $n = j$, then taking n derivatives of x^j leaves just $n!$, which is $n!$ when evaluated at 0.

Now, consider the odd derivatives of $f(x)$:

$$\begin{aligned}
\left.\frac{\partial^{2n+1}}{\partial x^{2n+1}}\left(f(x)\right)\right|_0 &= \sum_{n=0}^{\infty} \frac{\left.\sum_{j=0}^{\infty}(-1)^j \cdot \frac{\partial^{2n+1}}{\partial x^{2n+1}}\left(\frac{(2^k x)^{2j}}{(2j)!}\right)\right|_0}{k!} \\
&= \sum_{n=0}^{\infty} \frac{\left.\sum_{j=0}^{\infty}(-1)^j \cdot \frac{\partial^{2n+1}}{\partial x^{2n+1}}\left(\frac{(2^k x)^{2j}}{(2j)!}\right)\right|_0}{k!}
\end{aligned}$$

Because all of the terms in the inner sum have even coefficients for x, they in particular never have degrees that match up with $2n+1$: consequently, by our earlier observation, the $2n+1$-th derivative of x^{2j} evaluated at 0 is always 0. As a result, we have that

$$\frac{\partial^{2n+1}}{\partial x^{2n+1}}\left(f(x)\right)\Big|_0 = 0, \ \forall n.$$

Finally, consider the even derivatives:

$$\begin{aligned}\frac{\partial^{2m}}{\partial x^{2m}}\left(f(x)\right)\Big|_0 &= \sum_{n=0}^{\infty} \frac{\sum_{j=0}^{\infty}(-1)^j \cdot \frac{\partial^{2m}}{\partial x^{2m}}\left(\frac{(2^k x)^{2j}}{(2j)!}\right)\Big|_0}{k!} \\ &= \sum_{n=0}^{\infty} \frac{\sum_{j=0}^{\infty}(-1)^j \cdot \frac{2^{2kj}}{(2j)!} \cdot \frac{\partial^{2m}}{\partial x^{2m}}\left(x^{2j}\right)\Big|_0}{k!}\end{aligned}$$

Recall, one last time, our earlier observation that the n-th derivative of x^j evaluated at 0 is 0, unless $j=n$, in which case it's $j!$. This tells us that the sum

$$\sum_{j=0}^{\infty}(-1)^j \cdot \frac{2^{2kj}}{(2j)!} \cdot \frac{\partial^{2m}}{\partial x^{2m}}\left(x^{2j}\right)\Big|_0$$

is equal to just its m-th term: i.e. that

$$\sum_{j=0}^{\infty}(-1)^j \cdot \frac{2^{2kj}}{(2j)!} \cdot \frac{\partial^{2m}}{\partial x^{2m}}\left(x^{2j}\right)\Big|_0 = (-1)^{2m} \cdot \frac{2^{2mk}}{(2m)!} \cdot (2m!).$$

Consequently, we have that

$$\begin{aligned}\frac{\partial^{2m}}{\partial x^{2m}}\left(f(x)\right)\Big|_0 &= \sum_{n=0}^{\infty} \frac{(-1)^{2m} \cdot \left(\frac{2^{2mk}}{(2m)!} \cdot (2m!)\right)}{k!} \\ &= (-1)^{2m} \sum_{n=0}^{\infty} \frac{2^{2km}}{k!} \\ &= (-1)^{2m} \sum_{n=0}^{\infty} \frac{(2^{2m})^k}{k!} \\ &= (-1)^{2m} e^{2^{2m}}\end{aligned}$$

Consequently, we have that the Taylor series for $f(x)$ is given by

$$\sum_{k=0}^{\infty}(-1)^k \frac{e^{2^{2k}}}{(2k)!} \cdot x^{2k}.$$

Does this series converge? Well: from looking at these terms, it's not even clear that they're diminishing in size! In fact, for $x \neq 0$, examine the ratio of two consecutive terms:

$$\begin{aligned}\frac{(-1)^{k+1}\frac{e^{2^{2k+2}}}{(2k+2)!}\cdot x^{2k+2}}{(-1)^{k}\frac{e^{2^{2k}}}{(2k)!}\cdot x^{2k}} &= -\frac{e^{2^{2k+2}}\cdot x^{2k+2}\cdot(2k)!}{e^{2^{2k}}x^{2k}(2k+2)!}\\ &= -\frac{e^{2^{2k+2}-2^{2k}}}{x^{-2}(2k+1)(2k+2)}\\ &= -\frac{e^{4\cdot 2^{2k}-2^{2k}}}{x^{-2}(2k+1)(2k+2)}\\ &= -\frac{e^{3\cdot 2^{2k}}}{x^{-2}(2k+1)(2k+2)}.\end{aligned}$$

This clearly flies off to $\pm\infty$ as $k \to \infty$; therefore, we have that the individual terms of our series fail to converge to 0 whenever $x \neq 0$. As this is a necessary condition for a series to converge, we have thus finally shown that our series fails to converge for any value of $x \neq 0$, which is exactly what we wanted to show! □

As the above example hopefully illustrates, there are occasionally functions whose Taylor series are of little help to us at all; this is why we check to make sure that functions have *convergent* Taylor series, and worry about things like the remainders (which can tell us how big the gap between the function and its Taylor series is!)

However, this is not to say that Taylor series are not useful things to study: quite the opposite, in fact! Consider this last example as an illustration for how Taylor series are sometimes the easiest way to set about solving a problem:

2.2. **Integrating the Gaussian Function.** When we examine an indefinite integral in a calculus class, there is a tendency to always assume that the integral will be "nice." In other words, if you were working through your Math 1 HW and were asked to find the integral

$$\int \frac{x^8 - 8x + 1}{x^2 - 1}dx,$$

you would probably assume that (1) this indefinite integral exists, and furthermore that (2) there is some "nice" way to write out what it is in terms of polynomials and exponentials and trig things.

Given that this is Math 8, however, you'd probably assume that there was a trap. Which is often true! Not for the above integral: that one you can solve with partial fractions (which we'll discuss in week 10, I think.) Consider, however, the following indefinite integral:

$$\int e^{-x^2/2}dx.$$

The function $e^{-x^2/2}$ is continuous and bounded on $\mathbb{R}$; consequently, it is integrable, and there is some primitive $F(x)$ of it such that $F(x) = e^{-x^2/2}$.

Remarkably, there is no way to write this $F(x)$ using only elementary functions. In other words, suppose that you're allowed to work with the polynomials, exponential, trigonometric functions, and all of their inverses: and you can compose and multiply and divide and sum as (finitely) many of these functions as you like. There is no way that you can construct $F(x)$ in this way; in this sense, $F(x)$ is as elemental as the trigonometric functions and e^x.

This, as you may well imagine, makes calculating its integral over a given region rather difficult. In fact, short of returning to our upper- and lower-sum bounds, we really haven't developed any machinery for tackling such a task. However, if we merely want to find a *good approximation* for this integral over some region, we are in luck! Taylor series, in particular, are remarkably useful for such a task, as we show here:

Question 2.3. *Approximate*

$$\int_{-1}^{1} e^{-x^2/2} dx$$

to within $\pm .01$ *of its actual value.*

Proof. Well: because

$$P_n(e^x, 0) = \sum_{k=0}^{n} \frac{x^k}{k!},$$

we can compose $-x^2/2$ with e^x's Taylor polynomial to get that

$$P_{2n}(e^{-x^2/2}, 0) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{2^k k!}.$$

Using this, we can write

$$e^{-x^2/2} = P_{2n}(e^{-x^2/2}, 0) + R_{2n}(e^{-x^2/2}, 0),$$

where the P_{2n}-part is something we can understand and the R_{2n}-thing is something we can make rather small!

Well, mostly. We can indeed use Taylor's theorem to notice that for some $c \in (0, x)$,

$$R_n(e^{-x^2/2}, 0) = \frac{\left.\frac{\partial^{n+1}}{\partial x^{n+1}} \left(e^{-x^2/2}\right)\right|_c}{(n+1)!} x^{n+1};$$

however, as it turns out, it's remarkably difficult to come up with a closed form for the derivatives[1] of $e^{-x^2/2}$. By hand, however, it's not too hard to calculate a few

[1] It is possible, though: check out the Wikipedia article on the Hermite polynomials for some remarkable identities

of these derivatives:

$$\begin{aligned}
\frac{\partial^1}{\partial x^1}\left(e^{-x^2/2}\right) &= -x\left(e^{-x^2/2}\right)\\
\frac{\partial^2}{\partial x^2}\left(e^{-x^2/2}\right) &= \left(x^2-1\right)\left(e^{-x^2/2}\right)\\
\frac{\partial^3}{\partial x^3}\left(e^{-x^2/2}\right) &= \left(-x^3+3x\right)\left(e^{-x^2/2}\right)\\
\frac{\partial^4}{\partial x^4}\left(e^{-x^2/2}\right) &= \left(x^4-6x^2+3\right)\left(e^{-x^2/2}\right)\\
\frac{\partial^5}{\partial x^5}\left(e^{-x^2/2}\right) &= \left(-x^5+10x^3-15x\right)\left(e^{-x^2/2}\right)\\
\frac{\partial^6}{\partial x^6}\left(e^{-x^2/2}\right) &= \left(x^6-15x^4+45x^2-15\right)\left(e^{-x^2/2}\right)\\
\frac{\partial^7}{\partial x^7}\left(e^{-x^2/2}\right) &= \left(-x^7+21x^5-105x^3+105x\right)\left(e^{-x^2/2}\right)
\end{aligned}$$

The seventh derivative should suffice for our purposes. In this case, we have that

$$R_6(e^{-x^2/2},0) = \frac{\frac{\partial^7}{\partial x^7}\left(e^{-x^2/2}\right)\Big|_c}{(7)!}x^7$$
$$= \frac{\left(-c^7+21c^5-105c^3+105c\right)\left(e^{-c^2/2}\right)}{(7)!}x^7.$$

Simple graphical analysis / looking at minima and maxima tells us that the polynomial portion of $\frac{\partial^7}{\partial x^7}\left(e^{-x^2/2}\right)$ is bounded by ± 50 on $[-1,1]$. We can then trivially bound $e^{-x^2/2}$ by 1 as well, to get that

$$\left|R_6(e^{-x^2/2},0)\right| \le \frac{50}{(7)!}x^7 = \frac{50}{5040}x^7 \le \frac{x^7}{100}.$$

Write $\int_{-1}^{1} e^{-x^2/2}dx = \int_{-1}^{1} P_6(e^{-x^2/2},0)dx + \int_{-1}^{1} R_6(e^{-x^2/2},0)dx$. Then, we have

$$\begin{aligned}
\int_{-1}^{1} P_6(e^{-x^2/2},0)dx &= \int_{-1}^{1}\left(1-\frac{x^2}{2}+\frac{x^4}{8}-\frac{x^6}{48}\right)dx\\
&= \left(x-\frac{x^3}{6}+\frac{x^5}{40}-\frac{x^7}{336}\right)\Big|_{-1}^{1}\\
&= 2-\frac{2}{6}+\frac{2}{40}-\frac{2}{336}\\
&= \frac{2872}{1680}.
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{-1}^{1} R_6(e^{-x^2/2}, 0) dx\right| &\leq \int_{-1}^{1} \left|R_6(e^{-x^2/2}, 0)\right| dx \\
&\leq \int_{-1}^{1} \left|\frac{x^7}{100}\right| dx \\
&= \int_{-1}^{1} \frac{|x|^7}{100} dx \\
&= 2 \cdot \int_{0}^{1} \frac{x^7}{100} dx \\
&= 2 \cdot \left(\frac{x^8}{800}\right) \Big|_0^1 \\
&= 2 \cdot \left(\frac{x^8}{800}\right) \Big|_0^1 \\
&= \frac{1}{400}.
\end{aligned}$$

Combining, we have that

$$\int_{-1}^{1} e^{-x^2/2} dx = \frac{2872}{1680},$$

plus or minus $\frac{1}{400}$. □