MATH 8, SECTION 1, WEEK 8 - RECITATION NOTES

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Abstract. These are the notes from Wednesday, Nov. 17th's lecture, where we discussed logarithmic differentiation, L'Hôpital's rule, and some of the properties of e^x.

1. Random Question

Question 1.1. *You and 99 of your friends have just been teleported back in time to Soviet Russia, whereupon the KGB has promptly arrested all of you. You are all to be placed in separate cells in a prison in the middle of Siberia; there are no other prisoners in your prison, nor any guards other than the prison's warden, nor any way to escape or communicate with anyone in any other cells.*

In fact, the only thing that will happen in your lives that involves interaction with other people is the following: At random intervals during the day, the warden will call up a random prisoner into their office. There are only two things in the office which you, as a prisoner, can touch or get at:

- *A lightswitch. It doesn't seem to control anything, but you can flick it either up or down, and no one but the prisoners is able to touch it.*
- *A large, red button. If you press the red button before every prisoner has been in to see the warden at least once, the entire prison explodes and you all die (so don't do that.) However, if you press the red button after every prisoner has seen the warden at least once, then you all get to go free!*

Before you arrive at the prison, your KGB handlers explain this situation to you all as a group, and leave you alone to come up with a plan. Can you collectively come up with a way to ensure that, given enough time, you'll always escape from the prison?

A slightly easier question: if you like, you can assume that your KGB handlers mention that the light switch starts in the up position – this isn't necessary, but it helps. A bit.

Today's lecture is kind-of three miniature lectures at once: they're not entirely too related, beyond their general nature as calculus questions, but they are kind of cool.

2. Logarithmic Differentiation

As you may have noticed in this course, the logarithm and exponential functions show up everywhere in calculus; they have a remarkable variety of applications to problems, one of which is the process of **logarithmic differentiation**, which it is perhaps easiest to illustrate with an example:

Example 2.1. Calculate

$$(x^x)'.$$

Proof. We first note, as a warning, that techniques like the power rule ($(x^n)' = nx^{n-1}$) are completely useless here, as it only works for x raised to some constant power – not a variable!

What can we do? Well: what happens if we take the log of x^x? We get $x \cdot \ln(x)$, which is something we do know how do take a derivative of. Can we use this to take a derivative of x^x itself?

As it turns out, yes! Specifically, note that by applying the chain rule on the left and the product rule on the right, we have

$$\begin{aligned}
&(\ln(x^x))' = (x \cdot \ln(x))' \\
\Rightarrow\ &\frac{1}{x^x} \cdot (x^x)' = x \cdot \frac{1}{x} + 1 \cdot \ln(x) \\
\Rightarrow\ &\frac{1}{x^x} \cdot (x^x)' = 1 + \ln(x) \\
\Rightarrow\ &\qquad (x^x)' = x^x \cdot (1 + \ln(x))
\end{aligned}$$

□

As it turns out, we can pretty much always use the trick above to take derivatives. Specifically, suppose that we have some function of the form

$$f(x)^{g(x)}.$$

Then, if we take the log of this function, we have

$$\ln\left(f(x)^{g(x)}\right) = g(x) \cdot \ln(f(x));$$

if we then take derivatives of both sides using the chain and product rules, we finally have

$$\begin{aligned}
&\left(\ln\left(f(x)^{g(x)}\right)\right)' = (g(x) \cdot \ln(f(x)))' \\
\Rightarrow\ &\frac{1}{f(x)^{g(x)}} \cdot \left(f(x)^{g(x)}\right)' = \frac{g(x)}{f(x)} \cdot f'(x) + g'(x) \cdot \ln(f(x)) \\
\Rightarrow\ &\qquad \left(f(x)^{g(x)}\right)' = \left(\frac{g(x)}{f(x)} \cdot f'(x) + g'(x) \cdot \ln(f(x))\right) \cdot f(x)^{g(x)},
\end{aligned}$$

which is a general formula for $f(x)^{g(x)}$ that we can calculate using only the derivatives of $f(x)$ and $g(x)$. Convenient, right?

To illustrate this once more, we study one last example:

Example 2.2. Calculate

$$\left(\sin(x)^{\sin(x)}\right)'.$$

Proof. Simply plug in the formula we established above, for $f(x) = g(x) = \sin(x)$:

$$\begin{aligned}\left(\sin(x)^{\sin(x)}\right)' &= \left(\frac{g(x)}{f(x)} \cdot f'(x) + g'(x) \cdot \ln(f(x))\right) \cdot f(x)^{g(x)} \\ &= \left(\frac{\sin(x)}{\sin(x)} \cdot \cos(x) + \cos(x) \cdot \ln(\sin(x))\right) \cdot \sin(x)^{\sin(x)} \\ &= \sin(x)^{\sin(x)} \cos(x) \left(1 + \ln(\sin(x))\right).\end{aligned}$$

□

3. L'Hôpital's Rule

4. The Uniqueness of e^x

So: in class, we discussed a few properties of the exp function: specifically, we said that

- e is the unique number such that $\exp(1) = e$.
- $\exp(x + y) = \exp(x)\exp(y)$
- $(\exp(x))' = \exp(x)$.

We also claimed that $\exp(x) = e^x$, for any $x \in \mathbb{R}$: we prove this claim here, briefly.

Lemma 4.1. $\exp(x) = e^x$, *for every* $x \in \mathbb{R}$.

Proof. The claim holds for $x = 1$ by assumption.

Then, because exp turns addition into multiplication, we know that for any n

$$\begin{aligned} e = \exp(n/n) &= \exp\left(\frac{1}{n} + \ldots + \frac{1}{n}\right) = \left(\exp\left(\frac{1}{n}\right)\right)^n \\ \Rightarrow e^{1/n} &= \exp\left(\frac{1}{n}\right).\end{aligned}$$

Again, we can use the observation that exp turns addition into multiplication to see that for any m, n we have

$$\exp(m/n) = \exp\left(\frac{1}{n} + \ldots + \frac{1}{n}\right) = \left(\exp\left(\frac{1}{n}\right)\right)^m = e^{m/n},$$

and thus that these two functions agree on all of the rationals. Consequently, because both functions are continuous on $\mathbb{R}$ and the rationals are dense in the real numbers, we know that these functions in fact agree everywhere: i.e. that $\exp(x) = e^x$, for every $x \in \mathbb{R}$. □

One question we might ask is the following: how many other functions are "like" e^x? In other words, how many functions have just the two properties below:

- f is differentiable on $\mathbb{R}$,
- $f(1) = e$, and
- $f(x + y) = f(x)f(y)$.

Well – the property $f(x+y) = f(x)f(y)$ seems like something that will be interesting to play with; so let's see what we can get from that. This says that, essentially, this function transforms addition into multiplication – so what happens when we put the additive identity, 0, into the function? We'd expect to get the multiplicative identity – and indeed, because

$$\begin{aligned} &f(0) = f(0+0) = f(0)f(0) \\ \Rightarrow &f(0) = 1 \text{ or } f(0) = 0. \end{aligned}$$

But if $f(0) = 0$, then $f(1) = f(0+1) = f(0)f(1) = 0 \cdot e = 0$ which contradicts our second property – so $f(0) = 1$! I.e. it sends the additive identity to the multiplicative identity.

As well, this property is interesting from a derivative-point of view, in that it tells us that (by using the definition of the derivative and the above realization that $f(0) = 1$)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h\to 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= f(x) \cdot f'(0). \end{aligned}$$

So: in other words, the derivative of f at any point is just a constant ($f'(0)$) times the value of f at that point! This is remarkably e^x-like.

In fact, it means that our function **is** e^x! To see this, let $c = f'(0)$, and look at the function $f(x)/e^{cx}$. If we take a derivative of this function, we get

$$\begin{aligned} \left(f(x) \cdot \frac{1}{e^{cx}}\right)' &= f'(x) \cdot \frac{1}{e^{cx}} + f(x) \cdot -c \cdot \frac{1}{e^{cx}} \\ &= f'(0) \cdot f(x) \cdot \frac{1}{e^{x}} - c \cdot f(x) \cdot \frac{1}{e^{cx}} \\ &= f'(0) \cdot f(x) \cdot \frac{1}{e^{x}} - f'(0) \cdot f(x) \cdot \frac{1}{e^{cx}} \\ &= 0. \end{aligned}$$

Therefore, this function $f(x)/e^{cx}$ is a constant: furthermore, because

$$f(0) \cdot \frac{1}{e^{c\cdot 0}} = 1 \cdot 1 = 1,$$

we know that this function is in fact the constant 1! In other words, $f(x) = e^{cx}$. Finally, because $f(1) = e = e^1$, we know that c must be equal to 1; thus, at last, we have that

$$f(x) = e^x.$$

Strange, right? We started by assuming almost nothing – that $f(x)$ was simply a differentiable function that did that cool addition-into-multiplication thing – and got that it was e^x! It's kind of remarkable that e^x is, in a sense, the *only* function that does such a simple thing.