MATH 8, SECTION 1, WEEK 8 - RECITATION NOTES

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Abstract. These are the notes from Friday, Nov. 19th's lecture. In this talk, we discuss the integral test.

1. Random Question

Question 1.1. *Consider the following game you can play on a $n \times n$ board made of 1×1 squares:*

1. *To start, mark some of the squares on the board as "infected."*
2. *If a square shares edges with at least two infected squares, mark it as infected as well.*
3. *Repeat (2) until no more squares are ever marked.*

How many squares do you need to infect at the start to insure that the whole board is eventually infected? Can you prove that your number is the smallest number needed?

2. The Integral Test: Statement

Today's lecture is fairly brief, as Math 1a's lecture today was shorter than normal. Here, we discuss the integral test, which says the following:

Theorem 2.1. *(Integral test:) If $f(x)$ is a positive and monotonically decreasing function, then*

$$\sum_{n=N}^{\infty} f(n) \text{ converges if and only if } \int_N^{\infty} f(x)dx \text{ converges.}$$

It bears noting that the conditions "positive and monotonically decreasing" are *extremely necessary* in the above definition: if you examine functions that aren't monotonically decreasing, you can run into things like

$$f(x) = \begin{cases} x, & x \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

which has the unfortunate property that

$$\sum_{n=N}^{\infty} f(n) = \sum_{n=N}^{\infty} n = \infty$$

while

$$\int_N^{\infty} f(x)dx = 0.$$

To illustrate this theorem's use, we turn to some examples:

3. The Integral Test: Examples

Question 3.1. *Does the series*

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

converge?

Proof. First, notice that because

$$\left(xe^{-x^2}\right)' = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} < 0, \forall x > 1,$$

we know that this function is decreasing on all of $[1, \infty)$. As well, it is trivially positive on $[1, \infty)$: so we can apply the integral test to see that this series converges iff the integral

$$\int_{n=1}^{\infty} xe^{-x^2} dx$$

converges.

But this is not too hard to show! – by using the u-substitution $u = x^2$, we have that

$$\int_{n=1}^{\infty} xe^{-x^2} dx = \int_{1}^{\infty} \frac{e^{-u}}{2} du = -\frac{e^{-u}}{2}\bigg|_{1}^{\infty} = \frac{e}{2},$$

and that in particular this integral converges. Therefore

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

must convege as well. □

Question 3.2. *Does the series*

$$\sum_{n=3}^{\infty} \frac{1}{(\ln(n))^{\ln(\ln(n))}}$$

converge?

Proof. First, notice that because $\ln(x)$ is an increasing function, so is $(\ln(x))^{\ln(\ln(x))}$; consequently, $\frac{1}{(\ln(x))^{\ln(\ln(x))}}$ is a decreasing function on $[3, \infty)$. As well, because $\ln(x)$ is positive whenever $x > 1$, we know that this function is positive on $[3, \infty)$: so we can apply the integral test to see that our sum converges iff the integral

$$\int_{n=3}^{\infty} \frac{1}{(\ln(x))^{\ln(\ln(x))}} dx$$

converges.

So: perform the u-substitution $u = \ln(x), x = e^u, dx = e^u du$ to see that

$$\begin{aligned}\int_{n=3}^{\infty} \frac{1}{(\ln(x))^{\ln(\ln(x))}} dx &= \int_{n=3}^{\infty} \frac{1}{u^{\ln(u)}} \cdot e^u du\\ &= \int_{n=3}^{\infty} \frac{1}{e^{(\ln(u))^2}} \cdot e^u du\\ &= \int_{n=3}^{\infty} e^{u-(\ln(u))^2} du.\end{aligned}$$

Finally, because

$$\lim_{u\to\infty} u - (\ln(u))^2 = \infty,$$

we know that $e^{u-(\ln(u))^2}$ increases without bound, and thus its integral cannot converge. So our sum does not converge, as well. □

Sometimes the integral test itself may not be applicable, but the idea of relating integrals and sums can still be used to show something converges! To see an example of this, look at our last problem:

Question 3.3. *Does the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

converge?

Proof. No. To see why, simply notice that for any $k > 1 \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} &\geq \sum_{n=k}^{\infty} \frac{1}{n^{1+1/n}}\\ &\geq \sum_{n=k}^{\infty} \frac{1}{n^{1+1/k}}\end{aligned}$$

Now, notice that because $\frac{1}{x^{1+1/k}}$ is a decreasing function, we know that the sum $\sum_{n=k}^{\infty} \frac{1}{n^{1+1/k}}$ is strictly larger than the integral $\int_k^{\infty} \frac{1}{x^{1+1/k}} dx$; consequently, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \geq &\int_k^{\infty} \frac{1}{x^{1+1/k}} dx\\ =&\frac{1}{x^{1/k}} \cdot (-k)\Bigg|_k^{\infty}\\ =&0 - \frac{-k}{k^{1/k}}\\ =&k^{k-1/k}\\ \geq&\sqrt{k}, \forall k > 2.\end{aligned}$$

Thus, we have that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \geq \sqrt{k}, \forall k > 2;$$

consequently, this sum must diverge. $\square$