MATH 8, SECTION 1, WEEK 7 - RECITATION NOTES

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Abstract. These are the notes from Wednesday, Nov. 10rd's lecture, where we discussed integration by parts and integration by substitution.

1. Random Question

Question 1.1. *Can you color the plane with the seven colors $\{R, O, Y, G, B, I, V\}$ – i.e. assign one of the colors $\{R, O, Y, G, B, I, V\}$ to every point in $\mathbb{R}^2$ – in such a way that any two points in the plane with the same color are never distance 1 apart?*

2. Methods of Integration: Statements and Philosophy

In our last class, we discussed the two fundamental theorems of calculus, which (roughly speaking) said that integration and derivation were "inverse" operations to one another. A natural question to ask, then, is the following:

Motivating Question 2.1. *For derivation, we had two central tools:*

- *the **chain rule**: i.e. for differentiable f, g, we have $(f(g(x))' = f'(g(x)) \cdot g'(x)$.*
- *the **product rule**: i.e. for differentiable f, g, we have $(f(x) \cdot g(x))' = f'(x)g(x) + g'(x)f(x)$.*

If we apply the fundamental theorems of calculus to these two rules, will we get a pair of "integral" theorems as well?

As it turns out: yes! Consider the following two theorems, which are direct consequences of the fundamental theorems of calculus and the chain/product rules:

Theorem 2.2. *(Integration by Parts – i.e. the "integral product rule:") If f, g are a pair of C^1 functions on $[a, b]$ – i.e they have continuous derivatives on $[a, b]$ – then we have*

$$\int_a^b f(x)g'(x) = f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx$$

Theorem 2.3. *(Integration by Substitution – i.e. the "integral chain rule:") If f is a continuous function on $g([a, b])$ and g is a C^1 functions on $[a, b]$, then we have*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx.$$

To illustrate how these two theorems are used, we work a series of examples:

3. Methods of Integration: Examples

Question 3.1. *What's*

$$\int_1^2 x^2 e^x dx \ ?$$

Proof. Looking at this problem, it doesn't seem like a substitution will be terribly useful: so, let's try to use integration by parts!

How do these kinds of proofs work? Well: what we want to do is look at the quantity we're integrating (in this case, x^2e^x,) and try to divide it into two parts – a "$f(x)$"-part and a "$g'(x)$" part – such that when we apply the relation $\int f(x)g'(x) = f(x)g(x) - \int g(x)f'(x)$, our expression gets simpler!

To ensure that our expression does in fact get simpler, we want to select our $f(x)$ and $g'(x)$ such that

(1) we can calculate the derivative $f'(x)$ of $f(x)$ and find a primitive $g(x)$ of $g'(x)$, so that either
(2) the derivative $f'(x)$ of $f(x)$ is **simpler** than the expression $f(x)$, or
(3) the integral $g(x)$ of $g'(x)$ is **simpler** than the expression $g'(x)$.

So: often, this means that you'll want to put quantities like polynomials or $\ln(x)$'s in the $f(x)$ spot, because taking derivatives of these things generally simplifies them. Conversely, things like e^x's or trig functions whose integrals you know are good choices for the integral spot, as they'll not get much more complex and their derivatives are generally no simpler.

Specifically: what should we choose here? Well, the integral of e^x is a particularly easy thing to calculate, as it's just e^x. As well, x^2 becomes much simpler after repeated derivation: consequently, we want to make the choices

$$\begin{array}{ll} f(x) = x^2 & g'(x) = e^x \\ f'(x) = 2x & g(x) = e^x, \end{array}$$

which then gives us that

$$\begin{aligned} \int_1^2 x^2 e^x dx &= f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)dx \\ &= x^2e^x\Big|_1^2 - \int_1^2 2xe^x dx. \end{aligned}$$

Another integral! Motivated by the same reasons as before, we attack this integral with integration by parts as well, setting

$$\begin{array}{ll} f(x) = 2x & g'(x) = e^x \\ f'(x) = 2 & g(x) = e^x. \end{array}$$

This then tells us that

$$\begin{aligned}
\int_1^2 x^2 e^x dx &= x^2 e^x\Big|_1^2 - \int_1^2 2xe^x dx \\
&= x^2 e^x\Big|_1^2 - \left(f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)dx \right) \\
&= x^2 e^x\Big|_1^2 - \left(2xe^x\Big|_1^2 - \int_1^2 2e^x dx \right) \\
&= x^2 e^x\Big|_1^2 - \left(2xe^x\Big|_1^2 - 2e^x\Big|_1^2 \right) \\
&= 4e^2 - e^1 - \left(4e^2 - 2e^1 - 2e^2 + 2e^1\right) \\
&= 2e^2 - e^1.
\end{aligned}$$

So we're done! □

Question 3.2. *What is*

$$\int_0^2 x^2 \sin(x^3) dx \ ?$$

Proof. How do we calculate such an integral? Direct methods seem unpromising, and using trig identities seems completely insane. What happens if we try substitution?

Well: our first question is the following: **what should we pick?** This is the only "hard" part about integration by substitution – making the right choice on what to substitute in. In most cases, what you want to do is to find the part of the integral that you don't know how to deal with – i.e. some sort of "obstruction." Then, try to make a substitution that (1) will remove that obstruction, usually such that (2) the derivative of this substitution is somewhere in your formula.

Here, for example, the term $\sin(x^3)$ is definitely an "obstruction" – we haven't developed any techniques for how to directly integrate such things. So, we make a substitution to make this simpler! In specific: Let $u = x^3$. This turns our term $\sin(x^3)$ into a $\sin(u)$, which is much easier to deal with Also, the derivative $du = 3x^2dx$ is (up to a constant) being multiplied by our original formula – so this substitution seems quite promising. In fact, if we calculate, we have that

$$\int_0^2 x^2 \sin(x^3) dx = \int_0^2 \sin(x^3) \cdot \frac{1}{3} \cdot 3x^2 dx = \int_0^8 \sin(u) \cdot \frac{1}{3} \cdot du$$

which is an integral we *can* calculate (it's $\frac{\sin(8)}{3}$.)

(Note that when we made our substitution, we also changed the bounds! Please, please, always change your bounds when you make a substitution!) □

Question 3.3. *What is*

$$\int_1^2 \ln(x))dx \ ?$$

Proof. At first glance, this looks unapproachable with either method that we know: there's only one term in there, and all of our methods depend on splitting things up into two terms!

...or is there only one term? Well: if you think of $\ln(x)$ as $1 \cdot \ln(x)$, then we can actually do something! Specifically, we want to take a derivatie of $\ln(x)$, to make it simpler: so we pick

$$\begin{array}{ll} f(x) = \ln(x) & g'(x) = 1 \\ f'(x) = 1/x & g(x) = x, \end{array}$$

which then gives us that

$$\begin{aligned} \int_1^2 \ln(x))dx &= x\ln(x)\Big|_1^2 - \int_1^2 x \cdot (1/x)dx \\ &= x\ln(x)\Big|_1^2 - x\Big|_1^2 \\ &= 2\ln(2) - 2. \end{aligned}$$

□

Question 3.4. *What is*

$$\int_0^1 \left(x^2+1\right)^{-3/2} \ ?$$

Proof. Again, at first glance, this looks unapproachable: there's still only one term in there, and the trick from last time clearly won't work. So what can we do?

Well: what if we tried substitution, but **put something in** as opposed to taking something out? I.e. in our earlier substitution proof, we started with an expression in the form

$$\int_a^b f(g(x))g'(x)dx$$

and turned it into one of the form

$$\int_{g(a)}^{g(b)} f(x)dx.$$

What if we try going the other way around? I.e.: we could use the fact that, while $\left(x^2+1\right)^{-3/2}$ is a rather complicated thing,

$$\begin{aligned} \left(\tan^2(x)+1\right)^{-3/2} &= \left(\frac{\sin^2(x)}{\cos^2(x)}+1\right)^{-3/2} \\ &= \left(\frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)}\right)^{-3/2} \\ &= \left(\frac{1}{\cos^2(x)}\right)^{-3/2} \\ &= \cos^3(x) \end{aligned}$$

really isn't!

Specifically: if we let

$$f(x) = \left(x^2+1\right)^{-3/2}, \quad \begin{array}{l} g(x) = \tan(x), \\ g'(x) = \frac{1}{\cos^2(x)}, \end{array}$$

we have that

$$\begin{aligned}
\int_0^1 \left(x^2+1\right)^{-3/2} dx &= \int_0^1 f(x)dx \\
&= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} f(g(x))g'(x)dx \\
&= \int_0^{\pi/4} \cos^3(x) \cdot \frac{1}{\cos^2(x)} dx \\
&= \int_0^{\pi/4} \cos(x)dx \\
&= \sin(x)\Big|_0^{\pi/4} dx \\
&= \sqrt{2}/2.
\end{aligned}$$

□