MATH 8, SECTION 1, WEEK 7 - RECITATION NOTES

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Abstract. These are the notes from Monday, Nov. 8th's lecture, where we discussed the fundamental theorems of calculus.

1. Random Question

Question 1.1. *Consider the following process:*

(1) *Start with any natural number n.*
(2) *If $n = 1$, stop.*
(3) *Otherwise, there are two cases:*
   - *n is odd and greater than 1. In this case, replace n with $3n + 1$, and return to step (2).*
   - *n is even. In this case, replace n with $n/2$, and return to step (2).*

Must this process always stop? How long does it take to stop on input 2^k? 3^k?

2. The Fundamental Theorems of Calculus: Theorems and Explanations

The Fundamental Theorems of Calculus, at first glance, seem like rather formidable statements: their title is set in All Caps!, their statements seem kind of ponderous, and in general they just seem like tricky things to understand and use. Luckily for us, however, these two theorems are actually really simple statements: at their heart, all that they say is that integration and derivation "undo" each other – i.e. that for continuous functions $f(x)$,

- (1st FTC) the derivative of the integral of $f(x)$ is $f(x)$, and
- (2nd FTC) the integral of the derivative of $f(x)$ is also pretty much just $f(x)$ (up to a constant term.)

Put another way, the two FTC's say that integration and derivation are in some sense inverse operations to each other! (This intuitive idea should be second nature to those of you who've been through a standard calculus class before, and first encountered the idea of the integral as a kind of "antiderivative.")

We state these two theorems here, and in the next section illustrate two of their uses:

Theorem 2.1. *(The First Fundamental Theorem of Calculus:) Let $[a, b]$ be some interval. If f is a bounded and integrable function over the interval $[a, x]$ for any $x \in [a, b]$, then the function*

$$A(x) := \int_a^x f(t)dt$$

exists for all $x \in [a, b]$. Furthermore, if $f(x)$ is continuous, the derivative of this function, $A'(x)$, is equal to $f(x)$.

In other words: for continuous functions $f(x)$, the integral of the derivative of $f(x)$ is just $f(x)$.

Theorem 2.2. *(The Second Fundamental Theorem of Calculus:) Let $[a, b]$ be some interval. Suppose that $f(x)$ is a function that has $\varphi(x)$ as its primitive*[1] *on $[a, b]$; as well, suppose that $f(x)$ is bounded and integrable on $[a, b]$. Then, we have that*

$$\int_a^b f(x)dx = \varphi(b) - \varphi(a).$$

In other words: for a bounded and integrable function $f(x)$, the derivative of the integral of $f(x)$ is just $f(x)$, up to some constant term (given by $f(a)$, say.)

3. Two Applications of the Fundamental Theorems of Calculus

The Fundamental Theorems of Calculus have a number of useful applications in calculus: we describe two of these applications here.

3.1. **Integrating via the Derivative.**

One particular use of the Second Fundamental Theorem of Calculus is that it allows us to turn our knowledge of the derivative into knowledge about the integral. Specifically, it tells us that if we have a function $f(x)$ and another function $\varphi(x)$ such that $\varphi'(x) = f(x)$ – i.e. knowledge of the derivative – that $\varphi(x)$ **is** the integral of $f(x)$, up to some constant term!

To illustrate what we're talking about, consider the following two examples:

Example 3.1.

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1}.$$

Proof. x^p is a continuous and bounded function on $[0, b]$, for any b; furthermore, we know that

$$\left(\frac{x^{p+1}}{p+1}\right)' = \frac{p+1}{p+1}x^p = x^p, \forall x,$$

so $\frac{x^{p+1}}{p+1}$ is a primitive of x^p.

Consequently, the second fundamental theorem of calculus tells us that

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1} - \frac{0}{p+1} = \frac{b^{p+1}}{p+1},$$

as claimed. □

To get an idea of the power of the fundamental theorems of calculus, recall that proving this fact directly last week took us a class and a half of difficult calculations; here, it took one blackboard and perhaps five minutes. Remarkable, right?

Example 3.2.

$$\int_a^b \cos(x)dx = \sin(b) - \sin(a)$$

[1] A function $f(x)$ has $\varphi(x)$ as its **primitive** on some interval $[a, b]$ iff $\varphi'(x) = f(x)$ on all of $[a, b]$.

Proof. Our proof here is almost identical in structure to the above proof. Note that $\cos(x)$ is a continuous and bounded function on $[a, b]$, for any a, b; furthermore, we know that

$$(\sin(x))' = \cos(x), \forall x,$$

so $\sin(x)$ is a primitive of $\cos(x)$.

Consequently, the second fundamental theorem of calculus tells us that

$$\int_0^b \cos(x)dx = \sin(b) - \sin(a),$$

as claimed. □

3.2. **Applying the Chain Rule in the Integral.** A second use of the FTC's is in working with the composition of integrals and functions. In other words, suppose that you have a function of the form

$$F(x) = \int_a^{g(x)} f(t)dt,$$

for $f(x)$ some continuous function. How can you take the derivative of this function $F(x)$? Without the fundamental theorems of calculus, we'd be lost – simply taking the derivative of the integral itself is a difficult thing without the FTC's, and dealing with the composition of the integral with the function $g(x)$ seems inordinately difficult. Yet, with the fundamental theorems of calculus, this becomes rather simple! In fact, just let

$$H(x) = \int_a^x f(t)dt.$$

Then we have that $F(x) = H(g(x))$; consequently, the chain rule says that

$$F'(x) = H'(g(x)) \cdot g'(x).$$

Now, use the First Fundamental Theorem of Calculus to see that $H'(x) = f(x)$, and thus that

$$F'(x) = f(g(x)) \cdot g'(x);$$

something we can easily calculate!

To illustrate this method, we work two examples below:

Example 3.3. Calculate the derivative of the function

$$F(x) = \int_0^{x^2} \sin(t)dt.$$

Proof. First, define the function $G(x)$ as

$$G(x) := \int_0^x \sin(t)dt.$$

By the fundamental theorem of calculus, we know that

$$G'(x) := \sin(x).$$

Thus, because $G(x^2) = F(x)$, we can just use the chain rule to see that

$$\begin{aligned}(F(x))' &= (G(x^2))' \\ &= 2x \cdot G'(x^2) \\ &= 2x \cdot \left(\int_0^x \sin(t)dt\right)' \Bigg|_{x^2} \\ &= 2x \cdot \sin(x^2).\end{aligned}$$

□

Example 3.4. Calculate the derivative of the function

$$F(x) = \int_{1/x}^x \frac{1}{t} dt,$$

whenever $t > 0$.

Proof. First, define the function $G(x)$ as

$$G(x) := \int_1^x \frac{1}{t} dt.$$

Then, by the fundamental theorem of calculus, we have that

$$G'(x) := 1/x.$$

So: note that

$$F(x) = \int_{1/x}^x \frac{1}{t} dt = \int_1^x \frac{1}{t} dt - \int_1^{1/x} \frac{1}{t} dt = G(x) - G(1/x).$$

(Note that we defined the function G here as an integral starting at 1, not 0! This is because the integral $\int_0^x \frac{1}{t} dt$ doesn't even exist whenever x is nonzero. So, when you use linearity of your integrals to split them apart, do be careful that you're not accidentally breaking your integral into parts that don't exist!)

Then, with this expression of $F(x) = G(x) - G(1/x)$, we can just proceed by the chain rule:

$$\begin{aligned}(F(x))' &= (G(x) - G(1/x))' \\ &= G'(x) - (-\frac{1}{x^2}) \cdot G'(1/x) \\ &= 1/x + \frac{1}{x^2} \cdot \frac{1}{1/x} \\ &= 2/x.\end{aligned}$$

□