MATH 8, SECTION 1, WEEK 7 - RECITATION NOTES

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Abstract. These are the notes from Friday, Nov. 12th's lecture. In this talk, we discuss how to integrate polar and parametric equations.

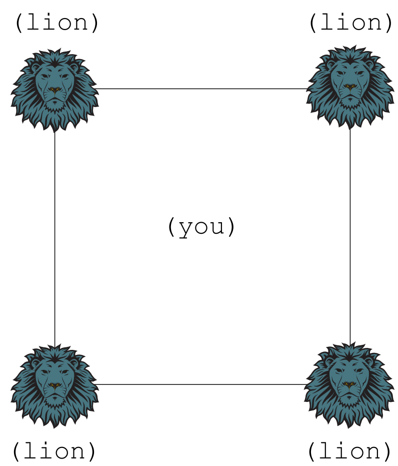
1. Random Question

Question 1.1. *You've been teleported back in time to ancient Rome, where (according to the fashion) you've been thrown into a gladiatorial pit. The pit is a perfect square, which you're in the exact center of: furthermore, at each of the four vertices of the square, there's a lion.*

The lions are restrained by chains so that they can only walk along the perimeter of the square: as well, both you and the lions

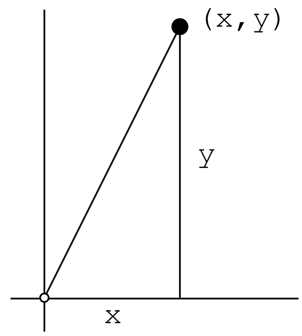
- *run at the same speed,*
- *are point-masses that can change direction arbitrarily quickly,*
- *are arbitrarily brilliant and can execute any given strategy, and*
- *know no fear.*

Can you escape the square – i.e. is there a strategy you can adopt that will always allow you to run out of the square without being devoured? What about other regular polygons: which of those can you escape? How?

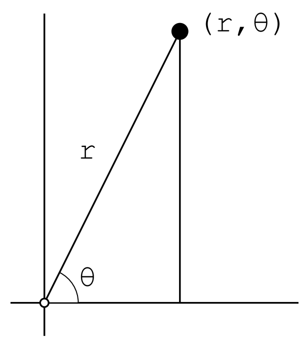


2. Polar Coördinates and Integration: Theory

Typically, when we discuss points in the plane, we use **Cartesian coördinates**, where we refer to a point in the plane as a pair (x, y), where x denotes its horizontal location alon the x-axis and y denotes its vertical location along the y-axis.



This, however, is not the only way to label the points in the plane! Specifically, another choice of coördinates you could make is that of the **polar coördinate system**, where we label every point in the plane as a pair (r, θ), where r denotes the distance of this point from the origin and θ denotes the angle made by the positive x-axis and the vector represented by our point, measured counterclockwise:

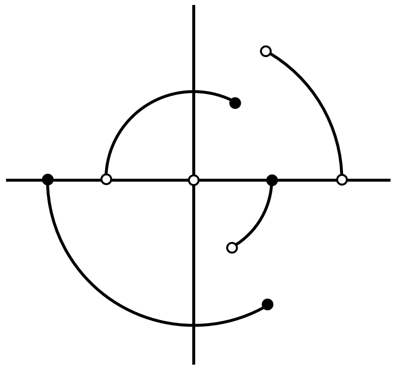


In this fashion, we can imagine the plane as the collection of all pairs of points $[0, 2\pi) \times \mathbb{R}^+$, and can study the graphs of functions $f(\theta)$ that map various angles to radial values. One immediate goal we might have, then, is the following: can we integrate these functions?

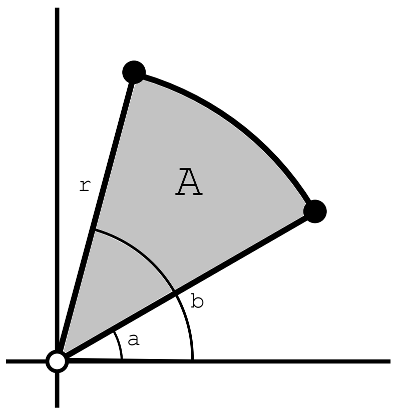
Well: for cartesian-coördinate-functions, we started by integrating step functions and used this to develop a theory of integration for the other functions. Can we do the same here?

First, let's say what a polar step function is:

Definition 2.1. A **polar step function** is a map $f : [a, b] \to \mathbb{R}$ with a partition $a = t_0 < t_1 < \ldots t_n = b$ of $[a, b]$, such that f is constant on each of the open intervals (t_{i-1}, t_i).



What is the area bounded by such a function? Well: first, recall that the area of a $(b-a)$-radian "wedge" of a circle with radius r is just $r^2 \cdot \frac{(b-a)}{2}$, as it's just $\frac{b-a}{2\pi}$-th of a circle with area πr^2.



Consequently, for a polar step function φ, we can say that the area radially bounded above by φ over $[a,b]$ is

$$\text{area}(\varphi,[a,b]) = \sum_{i=1}^{n} \left(\text{value}_{(t_{i-1},t_1)}(\varphi(\theta))\right)^2 \cdot \frac{(t_{i-1},t_i)}{2}$$

But this expression is just the "normal" integral – i.e. integral of φ as treated as a cartesian-coordinate-function– of $(\varphi(\theta))^2/2$: i.e.

$$\text{area}(\varphi,[a,b]) = \int_a^b \frac{\varphi(\theta))^2}{2} d\theta.$$

Does this formula hold in general? As it turns out, yes! Specifically, let's formally define the area radially bounded by a polar function as the following:

Definition 2.2. The area radially bounded above by a given polar function f on some interval $[a,b]$ is well-defined if and only if there are polar step functions L_n, U_n such that

- $L_n(\theta) \le f(\theta) \le U_n(\theta)$, for all $\theta \in [a,b]$
- $\lim_{n\to\infty} \text{area}(L_n,[a,b]) = \lim_{n\to\infty} \text{area}(U_n,[a,b])$.

In this case, we define

$$\text{area}(f, [a, b]) = \lim_{n\to\infty} \text{area}(L_n, [a, b]) = \lim_{n\to\infty} \text{area}(U_n, [a, b]).$$

Proposition 2.3. *If f is a polar function such that area$(f, [a, b])$ is well-defined, then*

$$\textit{area}(f, [a, b]) = \int_a^b \frac{(f(\theta))^2}{2} d\theta.$$

Proof. Take any such polar function f, and its associated lower and upper-bound-step functions L_n, U_n. Then, by definition, notice that we have

$$\lim_{n\to\infty} \text{area}(L_n, [a, b]) = \lim_{n\to\infty} \text{area}(U_n, [a, b]).$$
$$\Rightarrow \lim_{n\to\infty} \int_a^b \frac{(L_n(\theta))^2}{2} d\theta = \lim_{n\to\infty} \int_a^b \frac{(U_n(\theta))^2}{2} d\theta,$$

for some collection of lower and upper-bound step functions. But, because $L_n \leq f \leq U_n$, the definition of the normal integral tells us that

$$\int_a^b \frac{(f(\theta))^2}{2} d\theta$$

exists and is equal to both of the above limits! Consequently, we have that

$$\text{area}(f, [a, b]) = \int_a^b \frac{(f(\theta))^2}{2} d\theta,$$

as claimed □

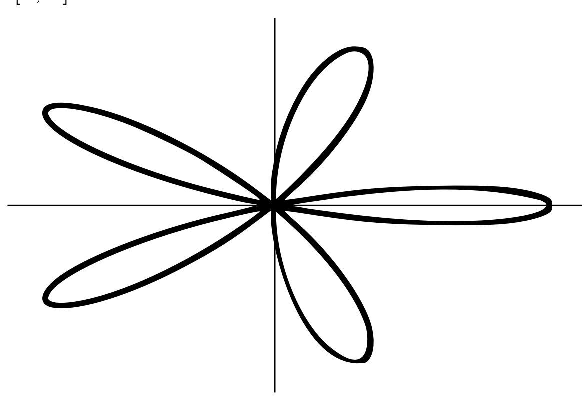
3. Polar Coördinates and Integration: Examples

To illustrate how these formulas are used, we work a few examples here.

Example 3.1. Find the area contained by the polar rose

$$f(\theta) = \cos(5\theta)$$

over the domain $[0, \pi]$.



Proof. By our above formula, we know that

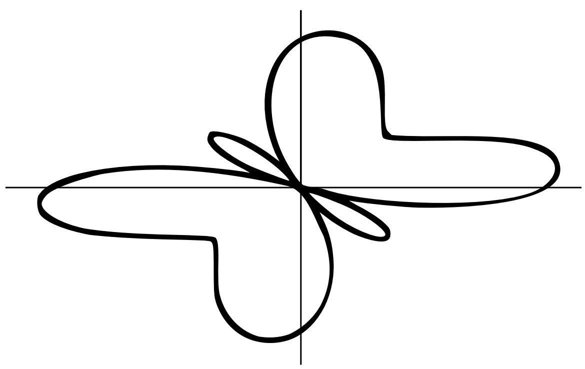
$$\begin{aligned}
\text{area(rose)} &= \int_0^{\pi} \frac{\cos^2(5\theta)}{2} d\theta \\
&= \int_0^{\pi} \frac{1+\cos(10\theta)}{4} d\theta \qquad\qquad \text{(double-angle formula)} \\
&= \left.\frac{10\theta + \sin(10\theta)}{40}\right|_0^{\pi} \\
&= \frac{10\pi + 0}{40} - \frac{10\cdot 0 + 0}{40} \\
&= \pi/4.
\end{aligned}$$

□

Example 3.2. Find the area contained within the "butterfly curve"

$$f(\theta) = 1 + \cos(4\theta) + \sin(2\theta)$$

over the domain $[0, 2\pi)$.



Proof. Again, we simply use our earlier formula and calculate, liberally using the double-angle formulas:

$$\begin{aligned}
\text{area(butterfly)} &= \int_0^{2\pi} \frac{(1+\cos(4\theta + \sin(2\theta))^2}{2} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(1 + \cos^2(4\theta) + \sin^2(2\theta) + 2\cos(4\theta) + 2\sin(2\theta) + 2\cos(4\theta)\sin(2\theta)\right) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \left(1 + \frac{1+\cos(8\theta)}{2} + \frac{1-\cos(4\theta)}{2} + 2\cos(4\theta) + 2\sin(2\theta) + 2\left(2\cos^2(2\theta) - 1\right)\sin(2\theta)\right) d\theta \\
&= \left.\left(\frac{\theta}{2} + \frac{8\theta + \sin(8\theta)}{32} + \frac{4 - \sin(4\theta)}{16} + \frac{\sin(4\theta)}{4} - \frac{\cos(2\theta)}{2}\right)\right|_0^{2\pi} + \int_0^{2\pi} \left(\left(2\cos^2(2\theta) - 1\right)\sin(2\theta)\right) d\theta \\
&= 2\pi + \int_0^{2\pi} \left(\left(2\cos^2(\theta) - 1\right)\sin(2\theta)\right) d\theta.
\end{aligned}$$

Finally, to evaluate the last integral, perform a u-substitution with $u = \cos(2\theta), u' = 2\sin(2\theta)$:

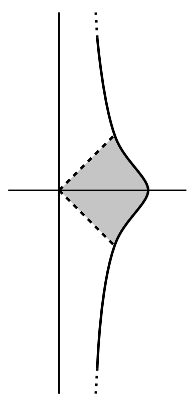
$$\begin{aligned}
\text{area(butterfly)} &= 2\pi + \int_0^{2\pi} \left(\left(2\cos^2(\theta) - 1\right)\sin(2\theta)\right) d\theta \\
&= 2\pi + \int_{\cos(0)}^{\cos(4\pi)} \frac{2u^2-1}{2} du \\
&= 2\pi + \int_1^1 \frac{2u^2-1}{2} du \\
&= 2\pi.
\end{aligned}$$

□

Example 3.3. Find the area bounded by the curve

$$f(\theta) = \frac{1}{\sqrt{\cos(\theta)}}$$

over the domain $[-\pi/4, \pi/4]$.



Proof. Once again, we simply apply our above formula:

$$\text{area}(f, [-\pi/4, \pi/4]) = \int_{-\pi/4}^{\pi/4} \frac{1}{2\cos(\theta)} d\theta.$$

How can we calculate this kind of integral? Well: as it's currently written, neither integration by parts nor integration by substitution seem to have any chance of working. However, if we try some algebra first, we can see that

$$\begin{aligned}
\frac{1}{2\cos(\theta)} &= \frac{1}{2}\frac{\cos(\theta)}{\cos^2(\theta)} \\
&= \frac{1}{2}\frac{\cos(\theta)}{1-\sin^2(\theta)} \\
&= \frac{1}{4}\left(\frac{\cos(\theta)}{1-\sin(\theta)} + \frac{\cos(\theta)}{1+\sin(\theta)}\right),
\end{aligned}$$

which we *can* integrate via substitution! Specifically, we have

$$\begin{aligned}
\text{area}(f, [-\pi/4, \pi/4]) &= \int_{-\pi/4}^{\pi/4} \frac{1}{2\cos(\theta)} d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{4} \left(\frac{\cos(\theta)}{1-\sin(\theta)} + \frac{\cos(\theta)}{1+\sin(\theta)} \right) d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{4} \cdot \frac{\cos(\theta)}{1-\sin(\theta)} d\theta + \int_{-\pi/4}^{\pi/4} \frac{1}{4} \cdot \frac{\cos(\theta)}{1+\sin(\theta)} d\theta.
\end{aligned}$$

Now, use the u-subsitution $u = 1 - \sin(\theta)$ in the first integral and $u = 1 + \sin(\theta)$ in the second:

$$\begin{aligned}
&\int_{-\pi/4}^{\pi/4} \frac{1}{4} \cdot \frac{\cos(\theta)}{1-\sin(\theta)} d\theta + \int_{-\pi/4}^{\pi/4} \frac{1}{4} \cdot \frac{\cos(\theta)}{1+\sin(\theta)} d\theta \\
&= \int_{1-\sin(-\pi/4)}^{1-\sin(\pi/4)} -1 \cdot \frac{1}{4} \cdot \frac{1}{u} du + \int_{1+\sin(-\pi/4)}^{1+\sin(\pi/4)} \frac{1}{4} \cdot \frac{1}{u} du \\
&= \int_{1+\sqrt{2}/2}^{1-\sqrt{2}/2} -1 \cdot \frac{1}{4} \cdot \frac{1}{u} du + \int_{1-\sqrt{2}/2}^{1+\sqrt{2}/2} \frac{1}{4} \cdot \frac{1}{u} du \\
&= \int_{1-\sqrt{2}/2}^{1+\sqrt{2}/2} \frac{1}{4} \cdot \frac{1}{u} du + \int_{1-\sqrt{2}/2}^{1+\sqrt{2}/2} \frac{1}{4} \cdot \frac{1}{u} du \\
&= 2 \int_{1-\sqrt{2}/2}^{1+\sqrt{2}/2} \frac{1}{4} \cdot \frac{1}{u} du \\
&= \frac{1}{2} \ln(u) \Big|_{1-\sqrt{2}/2}^{1+\sqrt{2}/2} \\
&= \frac{1}{2} \left(\ln\left(1 + \sqrt{2}/2\right) - \ln\left(1 - \sqrt{2}/2\right) \right) \\
&= \frac{1}{2} \ln \left(\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2} \right)
\end{aligned}$$

□

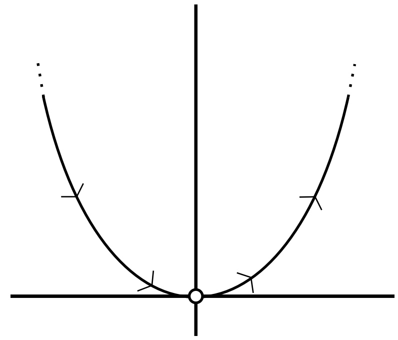
4. Parametrization and Integration: Theory

As we've discussed above, there are ways to draw curves in $\mathbb{R}^2$ other than using Cartesian coördinates: polar coördinates, as we've just discussed, provide a remarkably useful way to create simple curves. Are there other methods for drawing curves, that might perhaps also come in handy?

As it turns out, yes! Specifically, to draw a curve c in the plane, it suffices to offer a pair of functions $x(t)$ and $y(t) : \mathbb{R} \to \mathbb{R}$, and look at the graph of $c(t) = (x(t), y(t))$ on some interval $[a, b]$. For example, if we let

$$\begin{aligned}
x(t) &= t, \\
y(t) &= t^2,
\end{aligned}$$

we get the graph of the parabola sketched below:

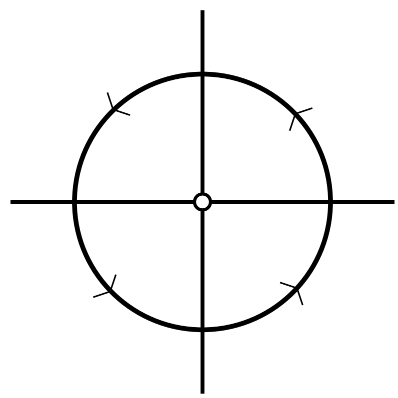


The arrows above indicate the path of the curve as t increases: i.e. small increases in the value of t follow the arrows drawn on the graph above.

Another example is the unit circle, given by the curve

$$x(t) = \cos(t),$$
$$y(t) = \sin(t),$$

on the interval $[0, 2\pi)$:



here, the curve starts at $(1, 0)$ and spirals counterclockwise, as drawn.

This process of describing a curve by a pair of equations is known as **parametrization**. A natural question to ask, then, is the following: can we come up with an expression for the "area" beneath a parametrized curve?

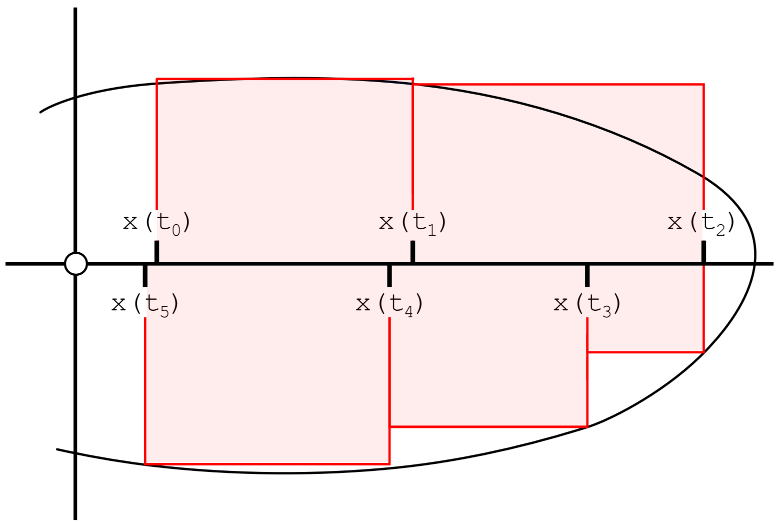
So: one idea we could use is the following:

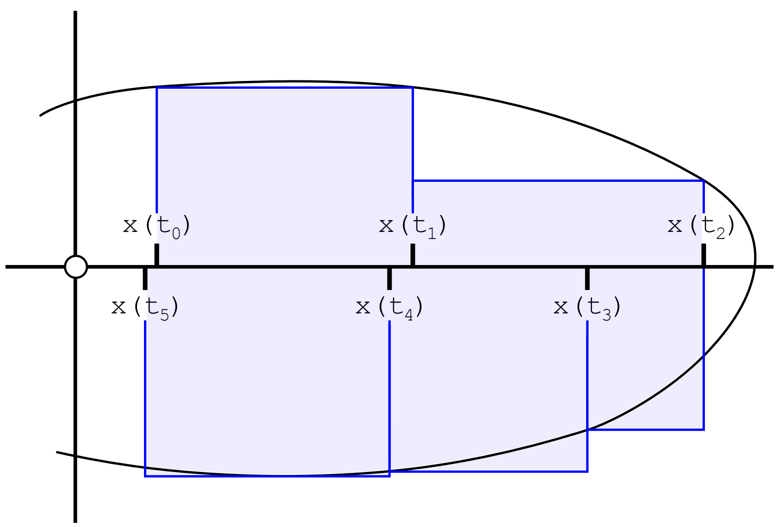
Definition 4.1. For a parametric curve $c(t) = (x(t), y(t))$, we say that the area beneath $c(t)$ on some interval $[a, b]$ is well-defined if and only if for every increasingly fine[1] set of partitions P_n of $[a, b]$, we have that

$$\lim_{n\to\infty}\left(\sum_{P_n} \sup_{(t_{i-1},t_i)} (y(t)) \cdot (x(t_i) - x(t_{i-1}))\right) = \lim_{n\to\infty}\left(\sum_{P_n} \inf_{(t_{i-1},t_i)} (y(t)) \cdot (x(t_i) - x(t_{i-1}))\right).$$

[1] A set of partitions is called **increasingly fine** if for every $\epsilon > 0$, there's a N so that for any $n > N$, we have that all of the distances $t_i - t_{i-1}$ are $< \epsilon$ in the partition $P_n = t_0 < \ldots < t_n$. In other words, increasingly fine partitions have arbitrarily tiny step sizes.

Pictorially, this just means that for any collection of incrasingly fine partitions P_n, the area of the red rectangles below (upper sums) converges to the area of the blue rectangles below (lower sums):





Given this definition, we can make the following observation:

Proposition 4.2. *If $c(t) = (x(t), y(t))$ is a parametrized curve such that the area beneath $c(t)$ on some interval $[a, b]$ is well-defined, $y(t)$ is continuous and bounded on $[a, b]$, and $x(t)$ is C^1 and bounded with bounded derivative on$[a, b]$, then*

$$area(c(t), [a, b]) = \int_a^b y(t)x'(t)dt.$$

Proof. By definition, we know that there are increasingly fine partitions P_n of $[a, b]$ such that

$$\lim_{n\to\infty}\left(\sum_{P_n}\sup_{(t_{i-1},t_i)}(y(t))\cdot(x(t_i)-x(t_{i-1}))\right)=\lim_{n\to\infty}\left(\sum_{P_n}\inf_{(t_{i-1},t_i)}(y(t))\cdot(x(t_i)-x(t_{i-1}))\right).$$

First: notice that we can rewrite

$$(x(t_i)-x(t_{i-1}))=\frac{(x(t_i)-x(t_{i-1}))}{(t_i-t_{i-1})}\cdot(t_i-t_{i-1}),$$

where the right-hand side looks a lot like the derivative of x times the length of (t_{i-1}, t_i). Specifically, we know that by the mean value theorem, there is some $c_i \in (t_{i-1}, t_i)$ such that

$$(x(t_i)-x(t_{i-1}))=x'(c_i)\cdot(t_i-t_{i-1}).$$

Furthermore, notice that we always have

$$\inf_{(t_{i-1},t_i)}(y(t))x'(c_i)\cdot(t_i-t_{i-1})\le y(c_i)\cdot x'(c_i)\cdot(t_i-t_{i-1})\le\sup_{(t_{i-1},t_i)}(y(t))x'(c_i)\cdot(t_i-t_{i-1}):$$

consequently, if we apply the squeeze theorem to the inf and sup limits above, we can see that both of these limits must also be equal to the limit

$$\lim_{n\to\infty}\left(\sum_{P_n}y(c_i)\cdot x'(c_i)\cdot(t_i-t_{i-1})\right).$$

But we also know that this limit is equal to the integral of $y(t)x'(t)$ from a to b, as it lies between the inf and sup sums for that function as well (and the integral $y(t)x'(t)$ exists because both functions are continuous and bounded on $[a, b]$.)

Thus, we have

$$\text{area}(c(t),[a,b])=\int_a^b y(t)x'(t)dt,$$

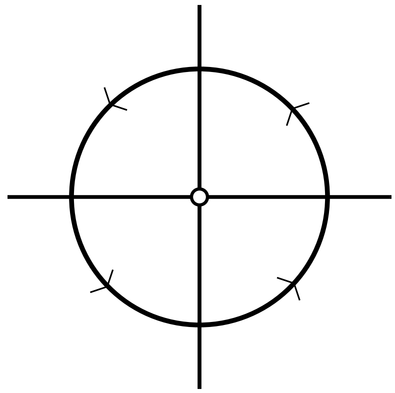
as claimed. □

The next section features several examples of how this formula works, and several caveats about its use:

5. Parametrization and Integration: Examples

Example 5.1. Find the unsigned area enclosed by the unit circle

$$c(t)=(\cos(t),\sin(t)).$$



Proof. If we apply this formula in a straightforward manner, we get

$$\begin{aligned}
\text{area}(c(t), [0, 2\pi]) &= \int_0^{2\pi} \sin(t) \cdot (-\sin(t))dt \\
&= -\int_0^{2\pi} \sin^2(t)dt \\
&= -\int_0^{2\pi} \frac{\cos(2t) - 1}{2}dt \\
&= \left.\frac{\sin(2t) - 2t}{4}\right|_0^{2\pi} \\
&= -\pi.
\end{aligned}$$

. . . wait. Where did that minus sign come from?

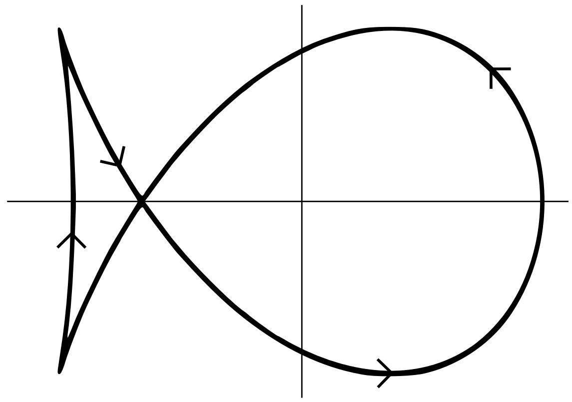
First: remember that our integrals are computing, typically, an idea of "signed area:" i.e. area beneath the x-axis is typically counted negatively, whereas area above the x-axis is counted positively. Well, as it turns out, the **orientation** of our parametrized curve *also* affects the sign of the area! Specifically, parametrized area is counted negatively when the curve's x-coördinates are decreasing, and positively when the curve's x-coördinates are increasing: you can see this in the derivation of our formula for the area of a parametric curve, as the sign of the sum

$$\lim_{n\to\infty} \left(\sum_{P_n} \sup_{(t_{i-1}, t_i)} (y(t)) \cdot (x(t_i) - x(t_{i-1})) \right)$$

depends both on the sign of $y(t)$ **and** of $(x(t_i) - x(t_{i-1}))$. So the curve above has *signed*area $-\pi$, but *unsigned* area π! This will become much more interesting in the next example: □

Example 5.2. Find the unsigned area enclosed by the ellipse negative pedal curve ("fish curve")

$$c(t) = \left(\cos(t) - \frac{\sin^2(t)}{\sqrt{2}}, \cos(t)\sin(t) \right).$$



Proof. If we calculate naively, we have that

$$\begin{aligned}
\text{area}(c(t),[0,2\pi]) &= \int_0^{2\pi} \cos(t)\sin(t)\cdot\left(-\sin(t) - \frac{2\sin(t)\cos(t)}{\sqrt{2}}\right)dt\\
&= \int_0^{2\pi} \left(-\cos(t)\sin^2(t) - \frac{2\sin^2(t)\cos^2(t)}{\sqrt{2}}\right)dt\\
&= \int_0^{2\pi} \left(-\cos(t)\sin^2(t) - \frac{\sin^2(2t)}{2\sqrt{2}}\right)dt\\
&= \int_0^{2\pi} \left(-\cos(t)\sin^2(t) - \frac{1-\cos(4t)}{4\sqrt{2}}\right)dt\\
&= \int_0^{2\pi} -\cos(t)\sin^2(t)dt - \int_0^{2\pi} \frac{1-\cos(4t)}{4\sqrt{2}}dt
\end{aligned}$$

Using the simple u-substitution $u = \sin(t)$ on the left integral and simply calculating for the right integral gives us that

$$\begin{aligned}
\text{area}(c(t),[0,2\pi]) &= \int_0^{2\pi} -\cos(t)\sin^2(t)dt - \int_0^{2\pi} \frac{1-\cos(4t)}{4\sqrt{2}}dt\\
&= \int_{\sin(0)}^{\sin(2\pi)} -u^2du - \int_0^{2\pi} \frac{1-\cos(4t)}{4\sqrt{2}}dt\\
&= -\frac{u^3}{3}\Bigg|_{\sin(0)}^{\sin(2\pi)} - \frac{4t-\sin(4t)}{16\sqrt{2}}\Bigg|_0^{2\pi}
\end{aligned}$$

which evaluates to

$$= -\frac{u^3}{3}\Bigg|_0^0 - \frac{4t-\sin(4t)}{16\sqrt{2}}\Bigg|_0^{2\pi} = \frac{-\pi}{2\sqrt{2}}.$$

You might think that this is the area of the above curve, just "signed negatively" because of the way that the curve is oriented: but is it?

No! In fact, if you examine the orientation of our curve, it turns out that our integral above has subtracted the area of the tail (which gets a positive sign due to the way that we traversed the curve) from the area of the body (which got a negative sign, again b/c of our curve's orientation.) So, to get the *real* area, what we should do is *sum* the area of the tail and the area of the body. To do this, we simply calculate each area separately, using the fact that the body is just our curve ran on $[-\pi/2, \pi/2]$ and that the tail is our curve ran from $\pi/2, 3\pi/2]$:

$$\begin{aligned}
\text{area(head)} &= -\frac{u^3}{3}\Big|_{\sin(-\pi/2)}^{\sin(\pi/2)} - \frac{4t - \sin(4t)}{16\sqrt{2}}\Big|_{-\pi/2}^{\pi/2} \\
&= -\frac{u^3}{3}\Big|_{-1}^{1} - \frac{4t - \sin(4t)}{16\sqrt{2}}\Big|_{-\pi/2}^{\pi/2} \\
&= -\frac{2}{3} - \frac{\pi}{4\sqrt{2}} \\
\text{area(tail)} &= -\frac{u^3}{3}\Big|_{\sin(\pi/2)}^{\sin(3\pi/2)} - \frac{4t - \sin(4t)}{16\sqrt{2}}\Big|_{\pi/2}^{3\pi/2} \\
&= -\frac{u^3}{3}\Big|_{1}^{-1} - \frac{4t - \sin(4t)}{16\sqrt{2}}\Big|_{\pi/2}^{3\pi/2} \\
&= \frac{2}{3} - \frac{\pi}{4\sqrt{2}}
\end{aligned}$$

Taking absolute values of the head and tail and then summing tells us that

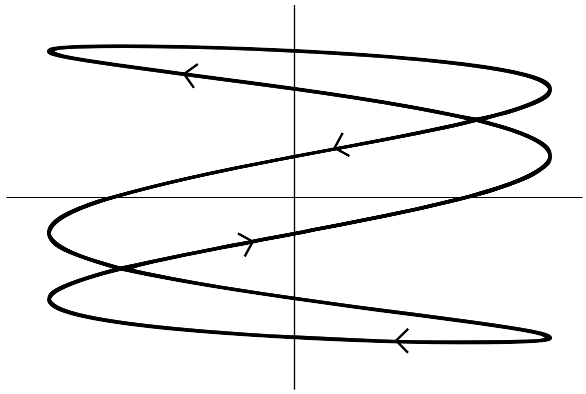
$$\begin{aligned}
\text{area(fish)} &= |\text{area(head)}| + |\text{area(tail)}| \\
&= \frac{2}{3} + \frac{\pi}{4\sqrt{2}} + \frac{2}{3} - \frac{\pi}{4\sqrt{2}} \\
&= \frac{4}{3},
\end{aligned}$$

which is in fact the actual area enclosed by our fish curve. □

We leave the last problem here as an exercise to the reader:

Example 5.3. Find the area enclosed by the Lissajous curve

$$c(t) = (\sin(3t + \pi/4), \sin(t)) .$$



Proof. □