MATH 8, SECTION 1, WEEK 6 - RECITATION NOTES

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Abstract. These are the notes from Monday, Nov. 1st's lecture, where we begin our discussion of the integral.

1. Random Question

Question 1.1. *Can you cover a* 10×10 *board with* 4×1 *dominoes, without any overlap?*

2. Integration: Definitions and Tools

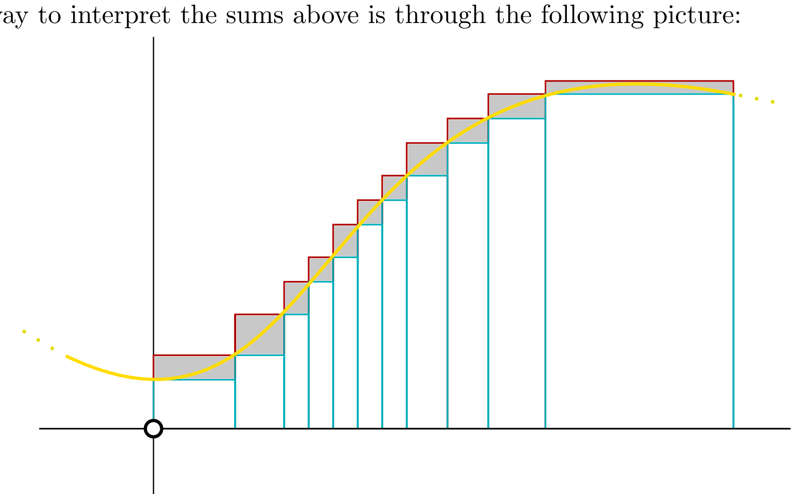
In class on Monday, we gave the following definition of the integral:

Definition 2.1. A function f is **integrable** on the interval $[a, b]$ if and only if the following holds:

- For any $\epsilon > 0$,
- there is a partition $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of the interval $[a, b]$ such that

$$\left(\sum_{i=1}^{n} \sup_{x \in (t_{i-1}, t_i)} (f(x)) \cdot \text{length}(t_{i-1}, t_i) - \sum_{i=1}^{n} \inf_{x \in (t_{i-1}, t_i)} (f(x)) \cdot \text{length}(t_{i-1}, t_i) \right) < \epsilon$$

One way to interpret the sums above is through the following picture:



Specifically,

- think of the $(\sum \inf)$-sum as the area of the blue rectangles in the picture below, and
- think of the $(\sum \sup)$-sum as the area of the red rectangles in the picture below.

- Then, the difference of these two sums can be thought of as the area of the gray-shaded rectangles in the picture above.
- Thus, we're saying that a function $f(x)$ is **integrable** iff we can find collections of red rectangles – an "upper limit" on the area under the curve of $f(x)$ – and collections of blue rectangles – a "lower limit" on the area under the curve of $f(x)$ – such that the area of these upper and lower approximations are arbitrarily close to each other.

Note that the above condition is equivalent to the following claim: if $f(x)$ is integrable, we can find a sequence of partitions $\{P_n\}$ such that "the area of the gray rectangles with respect to the P_n partitions goes to 0" – i.e. a sequence of partitions $\{P_n\}$ such that

$$\lim_{n\to\infty}\left(\sum_{P_n}\sup_{x\in(t_{i-1},t_i)}(f(x))\cdot \text{length}(t_{i-1},t_i)-\sum_{P_n}\inf_{x\in(t_{i-1},t_i)}(f(x))\cdot \text{length}(t_{i-1},t_i)\right)=0.$$

In other words, there's a series of partitions P_n such that these upper and lower sums both converge to the same value: i.e. a collection of partitions P_n such that

$$\lim_{n\to\infty}\left(\sum_{P_n}\sup_{x\in(t_{i-1},t_i)}(f(x))\cdot \text{length}(t_{i-1},t_i)\right)=\lim_{n\to\infty}\left(\sum_{P_n}\inf_{x\in(t_{i-1},t_i)}(f(x))\cdot \text{length}(t_{i-1},t_i)\right);$$

If this happens, then we define

$$\int_a^b f(x)dx=\lim_{n\to\infty}\left(\sum_{P_n}\sup_{x\in(t_{i-1},t_i)}(f(x))\cdot \text{length}(t_{i-1},t_i)\right)=\lim_{n\to\infty}\left(\sum_{P_n}\inf_{x\in(t_{i-1},t_i)}(f(x))\cdot \text{length}(t_{i-1},t_i)\right),$$

and say that this quantity is the **integral** of $f(x)$ on the interval $[a,b]$.

This discussion, hopefully, motivates why we often say that the integral of some function $f(x)$ is just "the area under the curve" of $f(x)$. Pictorially, we are saying that a function is integrable if and only if we can come up with a well-defined notion of area for this function; in other words, if sufficiently fine upper bounds for the area beneath the curve (the $(\sum\sup)$-sums) are arbitrarily close to sufficiently fine lower bounds for the area beneath the curve (the $(\sum\inf)$-sums.)

The definition of the integral, sadly, is a tricky one to work with: the sups and infs and sums over partitions amount to a ton of notation, and it's easy to get lost in the symbols and have no idea what you're actually manipulating. If you ever find yourself feeling confused in this way, just remember the picture above! Basically, there are three things to internalize about this definition:

- the area of the **red** rectangles corresponds to the upper-bound $(\sum\sup)$-sums,
- the area of the **blue** rectangles corresponds to the lower-bound $(\sum\inf)$-sums, and
- if these two sums can be made to be arbitrarily close to each other – i.e. the area of the gray rectangles can be made arbitrarily small – then we have a "good" idea of what the area under the curve is, and can say that $\int_a^b f(x)$ is just the limit of the area of those red rectangles under increasingly smaller partitions (which is also the limit of the area of the blue rectangles.)

While we've just introduced the integral, it does bear noting that we do have a pair of tools to help us calculate integrals:

Proposition 2.2. *If $f(x)$ is an integrable function on the interval $[a, b]$, and c is some point in $[a, b]$, then we have that*

$$\int_a^b f(x)dx = \int_a^x f(x)dx + \int_c^b f(x)dx.$$

Proposition 2.3. *Suppose that $f(x)$ is a step function*[1] *on the interval $[a, b]$, and $t_0 < t_1 < \ldots t_n$ is the corresponding partition of $[a, b]$ such that $f(x)$ is constant on each of the intervals (t_{i-1}, t_i). Then, we have that*

$$\int_a^b f(x)dx = \sum_{i=1}^{n} \mathit{value}_{(t_{i-1},t_i)}(f(x)) \cdot \mathit{length}(t_{i-1}, t_i).$$

3. Integration: An Example

Let's work an example here, to illustrate how to work with the integral:

Question 3.1. *What is the integral of the function*

$$f(x) = (-1)^p \cdot p, \text{ for } x \in [p-1, p)$$

from 0 to n, for any natural number n?

Proof. So: for any n, our function $f(x)$ is a step function on the interval $[0, n]$, with partition $0 < 1 < 2 < \ldots < n$ and function values $(-1)^p \cdot p$ on the intervals $(p-1, p)$. Therefore, we can use the earlier proposition to see that $f(x)$'s integral is

$$\begin{aligned}\int_0^n f(x)dx &= \sum_{p=1}^{n} \text{value}_{(p-1,p)}(f(x)) \cdot \text{length}(p-1, p)\\ &= \sum_{p=1}^{n} (-1)^p \cdot p\\ &= -1 + 2 - 3 + 4 \ldots + (-1)^n \cdot n.\end{aligned}$$

There are then two cases: if n is even, we have that

$$\begin{aligned}-1 + 2 - 3 + 4 \ldots + (-1)^n \cdot n &= (-1+2) + (-3+4) + \ldots + (-(n-1)+n)\\ &= 1 + 1 + \ldots + 1\\ &= \frac{n}{2},\end{aligned}$$

as we grouped together $n/2$ pairs of numbers in the first step.

Now, if n is odd, we instead have that

$$\begin{aligned}-1 + 2 - 3 + 4 \ldots + (-1)^n \cdot n &= -1 + (2 + -3) + (4 + -5) + \ldots + ((n-1) + -n)\\ &= -1 + -1 + \ldots + -1\\ &= -\frac{n+1}{2},\end{aligned}$$

[1] A function $f(x)$ is a **step function** on the interval $[a, b]$ if there is a partition $t_0 < t_1 < \ldots t_n$ of $[a, b]$ such that $f(x)$ is constant on each of the open intervals (t_{i-1}, t_i).

as there are $(n-1)/2$ pairs in the above grouping, along with the extra -1.

Combining our results, we have that

$$\int_0^n f(x)dx = \begin{cases} \frac{n}{2}, & n \text{ even}, \\ -\frac{n+1}{2}, & n \text{ odd}. \end{cases}$$

$\square$

(We started our discussion of the integral of x^p here; for the sake of continuity, this proof has been moved to the Wednesday notes. Go there!)