MATH 8, SECTION 1, WEEK 6 - RECITATION NOTES

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Abstract. These are the notes from Friday, Nov. 5th's lecture. In this talk, we discuss how to integrate discontinuous functions.

1. Random Question

Question 1.1. *A 3-**coloring** of a se X is a way to assign one of the three colors R, G, B to every point $x \in X$.*

Suppose that you have a 3-coloring of $\mathbb{R}^2$. Can you always find a pair of points in the plane such that

- *both of these points are the same color, and*
- *these two points are distance 1 apart from each other?*

2. Integrating Discontinuous Functions: The Theorem

In class today, Dr. Ramakrishnan finished his discussion of the following theorem:

Theorem 2.1. *If $f(x)$ is a bounded function on the interval $[a, b]$, and the collection of $f(x)$'s discontinuities on $[a, b]$ is a negligible set, then the integral*

$$\int_a^b f(x)dx$$

exists.

In this lecture, we will illustrate the use of this theorem. First, to do so, we should define the terms it uses: specifically, we should say what a **negligible set** is!

3. Negligible Sets

Definition 3.1. A set $X \subset \mathbb{R}$ is called **negligible**, or of measure 0, if the following holds: for any $\epsilon > 0$, there is some collection $\{I_n\}_{n=1}^{\infty}$ of closed intervals of positive length, such that

$$(1) \quad \bigcup_{n=1}^{\infty} I_n, \text{ the union of all of these intervals, } \supseteq X.$$

$$(2) \quad \sum_{n=1}^{\infty} \text{length}(I_n) \leq \epsilon.$$

In essence, sets are **negligible** if they don't take up any "space" on the real line – i.e. if we can cover them with intervals with arbitrarily small total length.

Given that definition, the sets that turn out to be negligible are rather surprising:

Claim 3.2. *Any finite set is negligible.*

Proof. Pick any finite set $X = \{x_1 \ldots x_n\}$, any $\epsilon > 0$, and let $I_k = (x_k - \epsilon/2n, x_k + \epsilon/2n)$. Then the union $\bigcup I_k$ contains all of the x_k's by definition; as well, because there are n total intervals and each interval has length $\leq \epsilon/n$, the total length of the I_k's is bounded by ϵ. Thus, this is a negligible set, as claimed. □

Claim 3.3. $\mathbb{N}$ *is negligible.*

Proof. Pick any $\epsilon > 0$, and let $I_k = (k - \epsilon/(2 \cdot 2^n), k + \epsilon/(2 \cdot 2^n))$. Then the union $\bigcup I_k$ contains $\mathbb{N}$ by definition; as well, we have that

$$\sum_{n=1}^{\infty} \text{length}(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon,$$

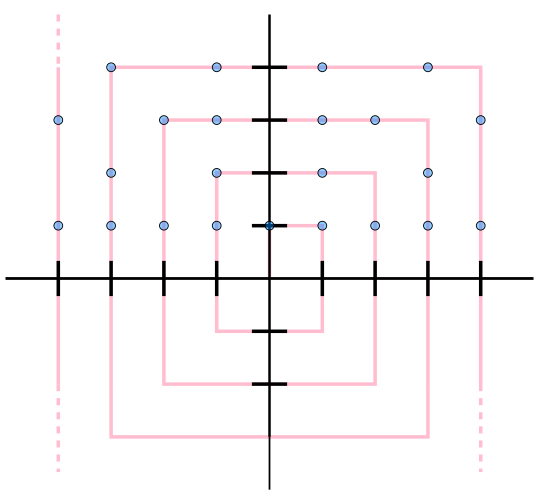
by using the geometric series $\sum_{n=1}^{\infty} 1/2^n = 1$. Therefore this set is negligible, as claimed. □

Claim 3.4. $\mathbb{Q}$ *is negligible.*

Proof. First, recall from week 2 the following observation, whose proof we briefly recap:

Lemma 3.5. *The sets* $\mathbb{N}$ *and* $\mathbb{Q}$ *are the same cardinality.*

Proof. Let $f : \mathbb{N} \to \mathbb{Q}$ be defined by setting $f(n)$ to be the n-th rational point found by starting at (0,0) and walking along the depicted spiral pattern below. This function hits every rational number exactly once by contstruction; thus, it is a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Consequently, $\mathbb{N}$ and $\mathbb{Q}$ are the same cardinality.



□

Consequently, we can write the rational numbers as some set indexed by the natural numbers – i.e. there's a way to write $\mathbb{Q}$ as some sequence $\{q_n\}_{n=1}^{\infty}$.

Pick any $\epsilon > 0$, and let $I_k = (q_k - \epsilon/(2 \cdot 2^n), q_k + \epsilon/(2 \cdot 2^n))$. Then the union $\bigcup I_k$ contains $\{q_n\}_{n=1}^{\infty} = \mathbb{Q}$ by definition; as well, we have that

$$\sum_{n=1}^{\infty} \text{length}(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon,$$

by using the geometric series $\sum_{n=1}^{\infty} 1/2^n = 1$. Therefore this set is negligible, as claimed. $\square$

It's worth taking a second to think about the weirdness of the above claim: the rational numbers are a set that is **dense** in the real line – they are, in a sense, everywhere. Yet, we've just shown that we can cover *all* of the rational numbers with intervals of arbitrarily small length! For example, there's a way to pick a bunch of closed intervals of positive length whose total length is just 1, and yet manage to contain *all* of the rational numbers in $\mathbb{Q}$! Crazy, right?

(It also bears noting that the proofs above can be easily adapted to show that any countable set is negligible, by using the same kinds of intervals.)

4. Integrating Discontinuous Functions: An Example

Now that we've got the terms of our theorem laid out, let's calculate an example:

Question 4.1. *Let*

$$f(x) = \begin{cases} (-1)^n, & x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \\ 0, & \textit{otherwise.} \end{cases}$$

Is $f(x)$ integrable on [0,1]? If so, what is its integral?

Proof. First, notice that $f(x)$ is discontinuous precisely at the points $\{1/2^n\}_{n=1}^{\infty}$ and 0 in $[0, 1]$; as this set is a subset of $\mathbb{Q}$, which is itself negligible, it is negligible. (Alternately, we can see that this set is negligible because it is countable.) Also, $f(x)$ is clearly bounded by ± 1; consequently, we can apply our theorem to see that $f(x)$ is integrable on $[0, 1]$.

So: what is its integral?

Well: we can't directly calculate it, as the only integrals we know how to calculate are those of step functions, and this isn't one of those. (It has infinitely many steps near 0, whereas we demanded that all of our step functions have a finite number of steps.) But we *can* do the following:

First, remember from Monday's class the following property of the integral:

Proposition 4.2. *If $f(x)$ is an integrable function on the interval $[a, b]$, and c is some point in $[a, b]$, then we have that*

$$\int_a^b f(x)dx = \int_a^x f(x)dx + \int_c^b f(x)dx.$$

Specifically, for our function $f(x)$, let $a = 0, c = 1/2^N$, and $b = 1$. Then, we have that for any $N \in \mathbb{N}$,

$$\int_0^1 f(x)dx = \int_0^{1/2^N} f(x)dx + \int_{1/2^N}^1 f(x)dx.$$

Then, on the interval $[1/2^N, 1]$, our function has only finitely many jumps; therefore, we can think of it as a step function! Consequently, we can calculate $\int_{1/2^N}^1 f(x)dx$. (for ease of calculation, we pick N to be even:)

$$\begin{aligned}
\int_{1/2^N}^1 f(x)dx &= \sum_{n=1}^{N} \text{value}_{\left(\frac{1}{2^{N-n}}, \frac{1}{2^{N-n-1}}\right)}(f(x)) \cdot \text{length}\left(\frac{1}{2^{N-n}}, \frac{1}{2^{N-n-1}}\right) \\
&= \sum_{n=1}^{N} \frac{(-1)^{N-n-1}}{2^{N-n}} \\
&= \sum_{n=1}^{N} \frac{(-1)^{n-1}}{2^{n}} \\
&= \left(1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{8} - 1 \cdot \frac{1}{16} + \ldots + \frac{(-1)^{N-1}}{2^n}\right) \\
&= \left(\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{8} - \frac{1}{16}\right) + \ldots + \left(\frac{1}{2^{N-1}} - \frac{1}{2^N}\right)\right) \\
&= \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots + \frac{1}{2^N}\right) \\
&= \sum_{n=1}^{N/2} \frac{1}{4^n}.
\end{aligned}$$

The integral $\int_0^{1/2^N} f(x)$ is harder to calculate, as it's not a step function: however, as it turns out, we don't have to actually find it! Instead, all we need to do is notice that because our theorem tells us that the integral $\int_0^{1/2^N} f(x)$ exists, that there is some series of partitions P_n of $[0, 1/2^N]$ such that

$$\int_0^{1/2^N} f(x) = \lim_{n\to\infty} \left(\sum_{P_n} \inf_{x \in (t_{i-1}, t_i)} (f(x)) \cdot \text{length}(t_{i-1}, t_i) \right).$$

But we know that $-1 \leq f(x) \leq 1$, for any $x \in [0, 1]$: consequently, we have that

$$\left(\sum_{P_n} -1 \cdot \text{length}(t_{i-1}, t_i)\right) \leq \left(\sum_{P_n} \inf_{x \in (t_{i-1}, t_i)} (f(x)) \cdot \text{length}(t_{i-1}, t_i)\right) \leq \left(\sum_{P_n} 1 \cdot \text{length}(t_{i-1}, t_i)\right).$$

But we know what the far-left and far-right sums are! They're just the length of the partition – $1/2^N$ – times ± 1. So, we have in fact that

$$-\frac{1}{2^N} \leq \int_0^{1/2^N} f(x) \leq \frac{1}{2^N}.$$

Combining our two results, we have that for any even natural number N,

$$\begin{aligned}
\int_0^1 f(x)dx &= \int_0^{1/2^N} f(x)dx + \int_{1/2^N}^1 f(x)dx \\
\Rightarrow \left|\int_0^1 f(x)dx\right| &\leq \left|\int_0^{1/2^N} f(x)dx + \int_{1/2^N}^1 f(x)dx\right| \\
&\leq \frac{1}{2^N} + \sum_{n=1}^{N/2} \frac{1}{4^n}.
\end{aligned}$$

Letting $N \to \infty$ then gives us

$$\begin{aligned}
\int_0^1 f(x)dx &= \lim_{N\to\ infty} \left(\frac{1}{2^N} + \sum_{n=1}^{N/2} \frac{1}{4^n} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{4^n} \\
&= \frac{1/4}{1 - 1/4} \\
&= \frac{1}{3}.
\end{aligned}$$

So we're done! $\square$