MATH 8, SECTION 1, WEEK 5 - RECITATION NOTES

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Abstract. These are the notes from Monday, Oct. 25th's lecture, where we begin our discussion of the derivative.

1. Random Question

Question 1.1. *Can you find a function* $f : [0,1] \to [0,1]$ *such that*

- $f(0) = 0, f(1) = 1,$
- f *is continuous, and*
- *everywhere* f *has a derivative, its derivative is 0?*

2. Differentiation: Definitions

Definition 2.1. For a function f defined on some neighborhood $(a-\delta, a+\delta)$, we say that f is **differentiable** at a iff the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a}$$

exists. If it does, denote this limit as $f'(a)$; we will often call this value the **derivative** of f at a.

The derivative has a number of interpretations as physical phenomena, which we will discuss in class on Wednesday; first, however, we will simply calculate a few derivatives to show how to attack these kinds of problems.

3. Differentiation: Examples

Lemma 3.1. *The derivative of* $f(x) = 1/x$ *at any point* $a \neq 0$ *is* $-1/a^2$.

Proof. Pick any point $a \neq 0$ in $\mathbb{R}$: then, a direct calculation of the limit tells us that

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a} &= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\
&= \lim_{h\to 0} \frac{\frac{a}{(a)\cdot(a+h)} - \frac{a+h}{(a)\cdot(a+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{a-a-h}{(a)\cdot(a+h)}}{h} \\
&= \lim_{h\to 0} \frac{-h}{(h)\cdot(a)\cdot(a+h)} \\
&= \lim_{h\to 0} \frac{-1}{(a)\cdot(a+h)}
\end{aligned}$$

But this is just the quotient of a pair of polynomials, the denominator of which is nonzero as $h \to 0$. Because polynomials are continuous, we then know that we can pass the limit through the quotient operation, and just evaluate the numerator and denominator's limits separately. Consequently, we have that this limit is $-1/a^2$, as claimed. $\square$

The above example was a rather quick and direct calculation; our second example, however, is a bit trickier:

Lemma 3.2. *The derivative of $f(x) = e^x$ at any point a is e^a.*

Before we begin this proof, however, we probably should define our terms: what do we even mean by e^x here, anyways?

3.1. **Power Series, e^x, and the Radius of Convergence: A Quick Detour.** In class around a week ago, we defined e^x as the following series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

One quick question to ask is this: why does this series even exist for every x? (Answer: ratio test! Try it if you're incredulous, or simply skip a few paragraphs to where we perform this calculation.)

Another, perhaps deeper question, is this: Suppose that we're working not with just $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, but some arbitrary series of the form $\sum_{n=0}^{\infty} a_n x^n$, where the a_n's are some sequence of constants. What happens then? Will these series always converge, just like e^x's series did? Will they sometimes not converge, for certain values of x?

To make this more specific: consider the following definition:

Definition 3.3. For a series of the form[1] $\sum_{n=0}^{\infty} a_n x^n$, we say that the **radius of convergence** of this series is some value $R \in \mathbb{R}$ such that

- if x is a real number such that $|x| < R$, $\sum_{n=0}^{\infty} a_n x^n$ converges, and
- if x is a real number such that $|x| > R$, $\sum_{n=0}^{\infty} a_n x^n$ diverges.

Our question now is the following: Does every power series have a radius of convergence? Are there power series that don't have $(-\infty, \infty)$ as their radius of convergence? Consider first the following three examples:

Example 3.4. The radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, as discussed earlier, is $(-\infty, \infty)$; this is because for any x,

$$\lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1,$$

and thus the ratio test tells us that this series converges, for any x.

The radius of convergence of $\sum_{n=0}^{\infty} \frac{|x|^n}{4^n}$ is $(-4, 4)$: this is because for any x,

$$\lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{4^{n+1}}}{\frac{|x|^n}{4^n}} = \lim_{n\to\infty} \frac{x}{4},$$

[1]Such series are called **power series**, because they are a series made out of increasing powers of x^n.

which is < 1 if $|x| < 4$, and > 1 if $|x| > 4$. Thus, we know by the ratio test that this series has radius of convergence $(-4, 4)$.

The radius of convergence of $\sum_{n=0}^{\infty} |x|^n \cdot n!$ is $\emptyset$. To see why, pick any $x \neq 0$, and notice that because the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converged, the limit as n approaches infinity of its individual terms $\frac{x^n}{n!}$ must be 0. Consequently, we know that the limit of the reciprocal of these terms,

$$\lim_{n\to\infty} \frac{n!}{x^n} = \lim_{n\to\infty} n! \cdot \left(\frac{1}{x}\right)^n$$

cannot exist and must fly off to infinity. But, if we write $y = 1/x$, this tells us that the limit of the individual terms in the series $\sum_{n=0}^{\infty} |y|^n \cdot n!$ doesn't exist – and thus is definitely not 0! Consequently, we know that this series cannot converge for any $x \neq 0$.

So: as it turns out, there are a number of fantastically useful properties about such "power series" and their radii of convergence. We'll return to these concepts later in the course, but for the purposes of exploring e^x we will state two of these properties here:

Proposition 3.5. *Every power series $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence.*

Proposition 3.6. *If $\sum_{n=0}^{\infty} a_n x^n$ is has the radius of convergence $(-R, R)$, then the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R, R)$.*

It's entirely within the scope of the tools you now have to prove both of these statements! However, in the interests of staying mostly on-topic, we will defer those proofs to a later date, and return to our original claim:

Lemma 3.7. *The derivative of $f(x) = e^x$ at any point a is e^a.*

Proof. We start by simply examining the derivative as a limit:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h) - a} &= \lim_{h\to 0} \frac{e^{a+h} - e^a}{h} \\
&= \lim_{h\to 0} \frac{e^a e^h - e^a}{h} \\
&= \lim_{h\to 0} e^a \cdot \frac{e^h - 1}{h} \\
&= \lim_{h\to 0} e^a \cdot \frac{\sum_{n=0}^{\infty} \frac{h^n}{n!} - 1}{h}
\end{aligned}$$

In order to bring the -1 and the division by h into our sum, we express it as a limit, and use the fact that limits play nicely with arithmetic:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a} &= \lim_{h\to 0} e^a \cdot \frac{\left(\lim_{n\to\infty} 1+\frac{h}{1}+\frac{h^2}{2!}+\ldots+\frac{h^n}{n!}\right)-1}{h}\\
&= \lim_{h\to 0} e^a \cdot \frac{\left(\lim_{n\to\infty} \frac{h}{1}+\frac{h^2}{2!}+\ldots+\frac{h^n}{n!}\right)}{h}\\
&= \lim_{h\to 0} e^a \cdot \left(\lim_{n\to\infty} \frac{1}{1}+\frac{h}{2!}+\ldots+\frac{h^{n-1}}{n!}\right)\\
&= \lim_{h\to 0} e^a \cdot \left(\sum_{n=1}^{\infty}\frac{h^{n-1}}{n!}\right)
\end{aligned}$$

This sum has radius of convergence $(-\infty,\infty)$, by the ratio test. Thus, if we think of it as a function with one variable h, it is continuous by our earlier observations on all of $\mathbb{R}$: consequently, the limit as $h\to 0$ of this series is just its value evaluated at 0: i.e.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a} &= e^a \cdot \left(\sum_{n=1}^{\infty}\frac{0^{n-1}}{n!}\right)\\
&= e^a\cdot 1\\
&= e^a.
\end{aligned}$$

Thus, the derivative of e^x at a is e^a, as claimed. □

4. Differentiation: Tools

As always, when we introduce a new definition, we like to create a number of tools to make using it a world easier. We list a few results below:

Proposition 4.1. *For f, g a pair of functions differentiable at a and α,β a pair of constants,*

$$(\alpha f(x)+\beta g(x))'(a)=\alpha f'(a)+\beta g'(a).$$

Proposition 4.2. *For f, g a pair of functions differentiable at a,*

$$(f(x)\cdot g(x))'(a)=f'(a)\cdot g(a)+g'(a)\cdot f(a).$$

Proposition 4.3. *For f a function differentiable at $g(a)$ and g a function differentiable at a,*

$$(f(g(x)))'(a)=f'(g(a))\cdot g'(a).$$

Basically, between these three theorems, most of your derivative calculations should come through the results we've established in class (how to take derivatives of trig functions, polynomials, and e^x) and just blind bashing with the above three rules.

To illustrate how this is done, consider the following example:

Lemma 4.4. *For any positive a,*

$$(\ln(x))'(a)=\frac{1}{a}.$$

Proof. To do this, we should maybe define what $\ln(x)$ is! So: define $\ln(x)$ as the function from $(0, \infty)$ to $\mathbb{R}$, that takes $x \in \mathbb{R}$ to the unique value $y \in \mathbb{R}$ such that $e^y = x$. Basically, $\ln(x)$ is the function that undoes e^x: it is the unique function such that $e^{ln(x)} = x$, for any positive x.

Using this definition, examine the quantity $e^{\ln(x)}$. On one hand, we know that this function is just x by definition; so

$$\left(e^{\ln(x)}\right)' = (x)' = 1.$$

On the other hand,

$$\begin{aligned}\left(e^{\ln(x)}\right)' &= e^{\ln(x)} \cdot (\ln(x))', \text{ (by the chain rule)} \\ &= x \cdot (\ln(x))'.\end{aligned}$$

Equating both sides, we have that

$$\begin{aligned}1 &= x \cdot (\ln(x))' \\ \Rightarrow \frac{1}{x} &= (\ln(x))'.\end{aligned}$$

So we're done! □