MATH 8, SECTION 1, WEEK 5 - RECITATION NOTES

TA: PADRAIC BARTLETT

Abstract. These are the notes from Friday, Oct. 29th's lecture. In this talk, we discuss the Weierstrass function, an example of a function that is continuous everywhere and differentiable nowhere.

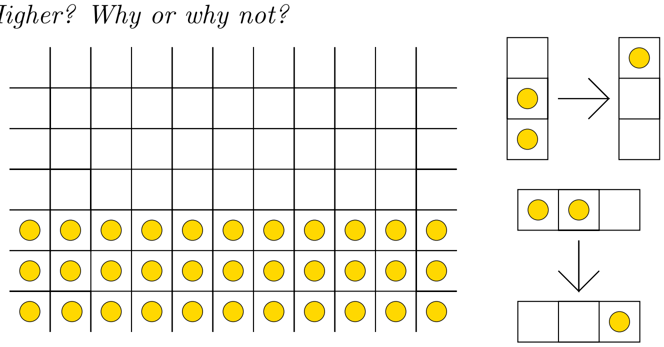
First: a warning: today's lecture was utterly ridiculous! Don't be afraid if it's confusing in parts, or complicated; the math going on here is really tricky! This result is typically presented in a graduate-level course in analysis; I wanted to show you guys that what you're doing in Math 1a is in fact real math, and that you can prove and study some seriously complicated things with the tools you have. So, if you're reviewing for a midterm, don't worry about this lecture too much! Think of this as a cross between an illustration of what you *can* do with your current tools, and a demonstration of a ton of proof techniques we've used/will use again throughout the course (with some aspects of an advertisement for advanced maths sprinked in!)

1. Random Question

Question 1.1. *Suppose you have a $\mathbb{Z} \times \mathbb{Z}$ grid of squares. Consider the following game we can play on this board:*

- *Starting position: place one coin on every single square below the x-axis.*
- *Moves: If there are two coins in a row (horizontally or vertically) with an empty space ahead of them, you can "jump" one of the coins over the other – i.e. you can remove those two coins and put a new coin on the space directly ahead of them.*

In this game, how "high" on the y-axis can you get a coin? Can you get one to height 3? Higher? Why or why not?



2. The Weierstrass Function

Today's lecture is centered around answering the following question:

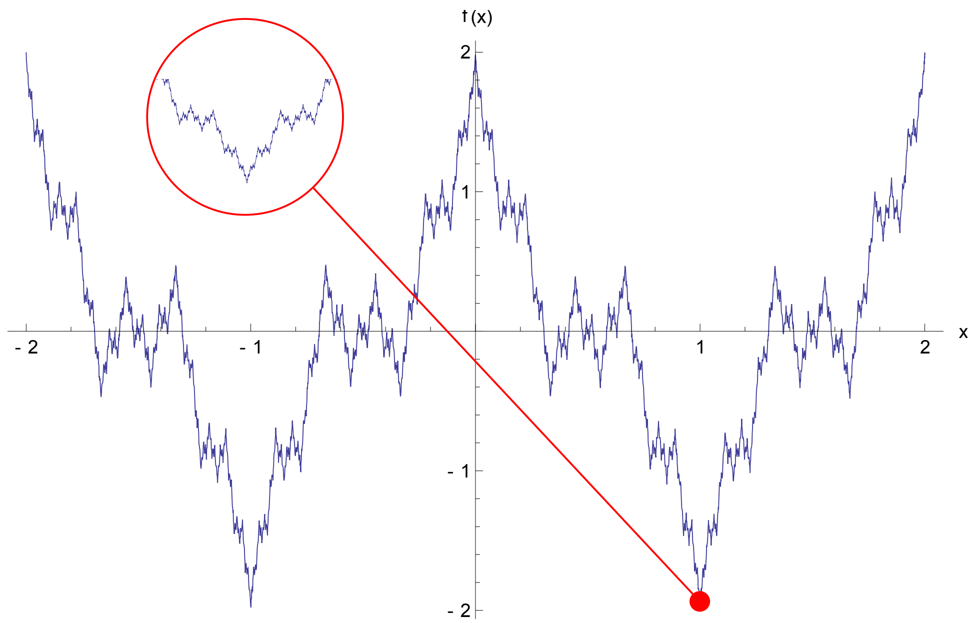
Question 2.1. *Is there a function that is*

- *continuous on all of* $\mathbb{R}$*, yet*
- *differentiable nowhere on* $\mathbb{R}$*?*

As it turns out, the answer is yes! Specifically, consider the following function:

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(101^n \cdot \pi x)}{2^n}.$$

Functions of this form (where 2 and 101 can be replaced with different constants a, b, that satisfy some set of properties) are called Weierstrass functions. A graph of a Weierstrass function, taken from Wikipedia, is shown below:



We will prove that our function has the claimed properties in three parts:

Claim 2.2. *The Weierstrass function* $f(x)$ *is defined – i.e. the series* $\sum_{n=1}^{\infty} \frac{\cos(101^n \cdot \pi x)}{2^n}$ *converges – for any value* $x \in \mathbb{R}$.

Proof. Because absolute convergence implies convergence, we know that it suffices to show that the series

$$\sum_{n=1}^{\infty} \frac{|\cos(101^n \cdot \pi x)|}{2^n}$$

converges. But this follows from a straightforward use of the comparison test with $\sum_{n=1}^{\infty} \frac{1}{2^n}$, because

$$\frac{|\cos(101^n \cdot \pi x)|}{2^n} \leq \frac{1}{2^n}, \forall x \in \mathbb{R}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent. □

Claim 2.3. *The Weierstrass function $f(x)$ is continuous at every $a \in \mathbb{R}$.*

Proof. Our strategy here is as follows:

- Because the series given by $f(x)$ converges at every $a \in \mathbb{R}$, its **tail** – i.e. the sum

$$\sum_{n=N+1}^{\infty} \frac{\cos(101^n \cdot \pi x)}{2^n},$$

for very large values of N – must be arbitrarily small!
- Thus, if we want to study $f(x)$, we can break it up into two pieces: the **head**, which is just the finite sum of continuous functions

$$\sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n},$$

and the tail, which is arbitarily small.
- So: specifically, if we're looking at $f(x) - f(a)$, we can break it up into

$$(\text{head of } f(x) - \text{head of } f(a)) + (\text{tail of } f(x)) + (\text{tail of } f(a)).$$

- As noted before, the tail parts are both arbitrarily small; so the only interesting thing is the difference of the head parts! But the head part is just the finite sum of continuous functions, and is thus itself continuous: thus, for values of x sufficiently close to a, the head part is also small! So this entire quantity is as small as we like: i.e. we can bound it by epsilon!

Basically, the rest of this proof is just following the plan laid out above! So if you don't care about details, feel free to skip the calculations below: but if you're curious, feel free to read on.

To begin our formal proof: choose any $\epsilon > 0$, and pick some large value of N such that $1/2^N \leq \epsilon/3$.

Then, examine the quantity $|f(x) - f(a)|$, and as suggested split our series into "head" and "tail" portions:

$$
\begin{aligned}
&|f(x) - f(a)| \\
&= \left| \sum_{n=1}^{\infty} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{\infty} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| \\
&= \left| \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} + \sum_{n=N+1}^{\infty} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi a)}{2^n} - \sum_{n=N+1}^{\infty} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| \\
&\leq \left| \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| + \left| \sum_{n=N+1}^{\infty} \frac{\cos(101^n \cdot \pi x)}{2^n} \right| + \left| \sum_{n=N+1}^{\infty} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| \\
&\leq \left| \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| + \sum_{n=N+1}^{\infty} \frac{|\cos(101^n \cdot \pi x)|}{2^n} + \sum_{n=N+1}^{\infty} \frac{|\cos(101^n \cdot \pi a)|}{2^n} \\
&\leq \left| \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| + \sum_{n=N+1}^{\infty} \frac{1}{2^n} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\
&\leq \left| \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| + \frac{1}{2^N} + \frac{1}{2^N} \\
&< \left| \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| + \frac{2}{3}\epsilon.
\end{aligned}
$$

Now: because

$$\sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n}$$

is a finite sum of continuous functions, it itself is continuous: therefore, for any $\epsilon/3$ and a there is some δ such that for all $|x - a| < \delta$,

$$\left| \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} \right| < \epsilon/3.$$

But this just tells us that for any such $|x - a| < \delta$,

$$|f(x) - f(a)| < \left| \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi x)}{2^n} - \sum_{n=1}^{N} \frac{\cos(101^n \cdot \pi a)}{2^n} \right| + \frac{2}{3}\epsilon < \frac{1}{3}\epsilon + \frac{2}{3}\epsilon = \epsilon.$$

As this is the definition for continuity at a point $a \in \mathbb{R}$, we can conclude that our function, $f(x)$, is continuous everywhere. □

Claim 2.4. *The Weierstrass function $f(x)$ is differentiable nowhere.*

Proof. We will use a similar approach to how we proved $f(x)$ was continuous. Specifically, we will try the following:

- By definition, the derivative of $f(x)$ at some point a is just

$$\begin{aligned}\lim_{h\to 0}\frac{f(h+a)-f(a)}{h} &= \lim_{h\to 0}\frac{\sum_{n=1}^{\infty}\frac{\cos(101^n\cdot\pi(h+a))}{2^n}-\sum_{n=1}^{\infty}\frac{\cos(101^n\cdot\pi(a))}{2^n}}{h}.\\ &= \lim_{h\to 0}\sum_{n=1}^{\infty}\frac{\cos\left(101^n\cdot\pi(h+a)\right)-\cos\left(101^n\cdot\pi(a)\right)}{h\cdot 2^n}.\end{aligned}$$

- Take this quantity and split it into a "head" portion and a "tail" portion, like we did before:

$$\begin{aligned}&\lim_{h\to 0}\sum_{n=1}^{\infty}\frac{\cos\left(101^n\cdot\pi(h+a)\right)-\cos\left(101^n\cdot\pi(a)\right)}{h\cdot 2^n}\\ &=\lim_{h\to 0}\left(\sum_{n=1}^{N-1}\frac{\cos\left(101^n\cdot\pi(h+a)\right)-\cos\left(101^n\cdot\pi(a)\right)}{h\cdot 2^n}+\sum_{n=N}^{\infty}\frac{\cos\left(101^n\cdot\pi(h+a)\right)-\cos\left(101^n\cdot\pi(a)\right)}{h\cdot 2^n}\right).\end{aligned}$$

  Denote the head portion of the above sum as H_N, and the tail portion of the above as T_N.
- Because

$$\lim_{h\to 0}\frac{f(h+a)-f(a)}{h}=\lim_{h\to 0}|H_N+T_N|\geq\lim_{h\to 0}|T_N|-|H_N|,$$

  if we can show that the "tail" part of our function grows in magnitude much faster than the head part – i.e. that $\lim_{h\to 0}|T_N|-|H_N|$ does not exist – then we have that the limit $\lim_{h\to 0}\frac{f(h+a)-f(a)}{h}$ also must diverge! In other words, our function would not be differentiable at a.

So, this is our basic strategy: to show that $|T_N|$ grows much faster than $|H_N|$. First, note the following useful lemma:

Lemma 2.5. *For any* $x, y\in\mathbb{R}$, $|\cos(x)-\cos(y)|\leq|x-y|$.

Proof. Simply apply the mean-value theorem to cos on $[x, y]$ to see that there is some $c\in[x, y]$ such that

$$\begin{aligned}&(\cos(x)-\cos(y))=-\sin(c)\cdot(x-y)\\ \Rightarrow&|\cos(x)-\cos(y)|=|\sin(c)|\cdot|x-y|\leq|x-y|.\end{aligned}$$

□

Let's use this lemma to create an upper bound for $|H_N|$:

$$\begin{aligned}
|H_N| &= \left| \sum_{n=1}^{N-1} \frac{\cos(101^n \cdot \pi(h+a)) - \cos(101^n \cdot \pi(a))}{h \cdot 2^n} \right|. \\
&\leq \sum_{n=1}^{N-1} \frac{|\cos(101^n \cdot \pi(h+a)) - \cos(101^n \cdot \pi(a))|}{|h| \cdot 2^n}. \\
&\leq \sum_{n=1}^{N-1} \frac{|(101^n \cdot \pi(h+a)) - (101^n \cdot \pi(a))|}{|h| \cdot 2^n}. \\
&= \sum_{n=1}^{N-1} \frac{|101^n \cdot \pi(h)|}{|h| \cdot 2^n}. \\
&= \pi \cdot \sum_{n=1}^{N-1} \left(\frac{101}{2}\right)^n \\
&= \pi \cdot \frac{(101/2)^N - 1}{(101/2) - 1} \\
&\leq \pi \cdot \frac{(101/2)^N}{(101/2) - 1} \\
&= \frac{2\pi}{99} \cdot \left(\frac{101}{2}\right)^N.
\end{aligned}$$

Ok: so that's an upper bound! Now, let's look at the tail portion, $|T_N|$. How can we control this?

Well: first, notice that for any N, we can write

$$101^N \cdot a = I_N + \xi_N,$$

where I_N is the closest integer to $101^N \cdot a$, and $-1/2 \leq \xi_N \leq 1/2$.

Let's examine what happens to $|T_N|$ when

$$h = \frac{1 - \xi_N}{101^N}.$$

In this situation, we have that for any $n \geq N$,

$$\begin{aligned}
101^N \cdot \pi \cdot (a+h) &= 101^{n-N} \cdot \pi \cdot \left(101^N a + 101^N h\right) \\
&= 101^{n-N} \cdot \pi \cdot \left(I_N + \xi_N + 101^N \cdot \frac{1-\xi_N}{101^N}\right) \\
&= 101^{n-N} \cdot \pi \cdot (I_N + 1),
\end{aligned}$$

an integer multiple of π that is odd if and only if $I_N + 1$ is odd. Thus, we have that

$$\cos(101^n \cdot \pi \cdot (a+h)) = (-1)^{I_N+1}, \text{ for any } n,$$

because cosine is equal to ± 1 on integer multiples of π.

By similar manipulations, we also have that

$$\begin{aligned}\cos\left(101^n\cdot\pi\cdot(a)\right) &= \cos\left(101^{n-N}\cdot\pi\cdot(101^N\cdot a)\right)\\ &= \cos\left(101^{n-N}\cdot\pi\cdot(I_N+\xi_N)\right)\\ &= \cos\left(101^{n-N}\cdot\pi\cdot(I_N)\right)\cdot\cos\left(101^{n-N}\cdot\pi\cdot(\xi_N)\right)\\ &= (-1)^{I_N}\cdot\cos\left(101^{n-N}\cdot\pi\cdot(\xi_N)\right)\end{aligned}$$

So: if we stick these two observations together and look at T_N, we have that

$$\begin{aligned}T_N &= \sum_{n=N}^{\infty}\frac{\cos\left(101^n\cdot\pi(h+a)\right)-\cos\left(101^n\cdot\pi(a)\right)}{h\cdot 2^n}\\ &= \sum_{n=N}^{\infty}\frac{(-1)^{I_N+1}-(-1)^{I_N}\cdot\cos\left(101^{n-N}\cdot\pi(\xi_N)\right)}{h\cdot 2^n}\\ &= (-1)^{I_N+1}\cdot\sum_{n=N}^{\infty}\frac{1+\cdot\cos\left(101^{n-N}\cdot\pi(\xi_N)\right)}{h\cdot 2^n}.\end{aligned}$$

Because all of the terms in the sum above are positive (as cosine always has magnitude ≤ 1,) we know that this sum is greater than just its first term! Consequently, we have that

$$\begin{aligned}|T_N| &\geq \left|\frac{1+\cos\left(101^{N-N}\cdot\pi(\xi_N)\right)}{h\cdot 2^N}\right|\\ &= \left|\frac{1+\cos\left(\pi\xi_N\right)}{h\cdot 2^N}\right|.\end{aligned}$$

Because $-1/2 \leq \xi_N \leq 1/2$, we know that $\cos(\pi\xi_N) \geq 0$, and thus that $1 + \cos(\pi\xi_N) \geq 1$; consequently, we have

$$\begin{aligned}|T_N| &\geq \left|\frac{1}{h\cdot 2^N}\right|\\ &= \frac{101^N}{2^N\cdot(1-\xi_N)}\\ &\geq \frac{2}{3}\cdot\left(\frac{101}{2}\right)^N,\end{aligned}$$

again because $-1/2 \leq \xi_N \leq 1/2$.

So: combining these two results tells us that at this value of h, we have

$$
\begin{aligned}
\frac{f(h+a)-f(a)}{h} &= |H_N + T_N| \\
&\geq |T_N| - |H_N| \\
&\geq \frac{2}{3}\cdot\left(\frac{101}{2}\right)^N - \frac{2\pi}{99}\cdot\left(\frac{101}{2}\right)^N \\
&\geq \left(\frac{2}{3}-\frac{2\pi}{99}\right)\cdot\left(\frac{101}{2}\right)^N.
\end{aligned}
$$

So: if we let $N \to \infty$, we have a sequence of values of h that converge to 0, on which $\frac{f(h+a)-f(a)}{h}$ diverges to infinity. Therefore, we know that the limit $\lim_{h\to 0}\frac{f(h+a)-f(a)}{h}$ cannot be any finite real number; consequently, we have that our function is not differentiable at a, for any $a \in \mathbb{R}$.

$\square$