# MATH 8, SECTION 1, WEEK 4 - RECITATION NOTES

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Abstract. These are the notes from Friday, Oct. 22nd's lecture. In this talk, we wrap up a number of loose ends relating to continuity and limits, discussing one-sided limits, limits at infinity, the intermediate value theorem, and the concepts of open, closed, and bounded sets.

## 1. Random Question

**Question 1.1.** *Can you find a function $f : [0,1] \to [0,1]$such that*

- $f$ *is continuous,*
- $f(0) = 0, f(1) = 1$*, and*
- $f$ *takes on every value in the interval* $(0,1)$ *exactly once? Twice? Three times? n times? Infinitely many times?*

Today's lecture is kind of a grab-bag of topics; where Monday and Wednesday's lectures were devoted to exploring a pair of complicated topics slowly and carefully, most of the ideas in today's lecture are relatively short and sweet. Consequently, we'll move at a faster pace; there are about four concepts that we should cover today, each of which is hopefully a little related to the others and should be useful in your study of limits and continuity.

## 2. One-Sided Limits

Let's start with something fairly elementary: the concept of a **one-sided limit**:

**Definition 2.1.** For a function $f : X \to Y$, we say that

$$\lim_{x \to a^+} f(x) = L$$

if and only if

(1) (vague:) as $x$ goes to $a$ from the right-hand-side, $f(x)$ goes to $L$.
(2) (concrete, symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x-a| < \delta \text{ and } x > a) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x \to a^-} f(x) = L$$

if and only if

(1) (vague:) as $x$ goes to $a$ from the left-hand-side, $f(x)$ goes to $L$.
(2) (concrete, symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x-a| < \delta \text{ and } x < a) \Rightarrow (|f(x) - L| < \epsilon).$$

Basically, this is just our original definition of a limit except we're only looking at $x$-values on one side of the limit point $a$: hence the name "one-sided limit." Thus, our methods for calculating these limits are pretty much identical to the methods we introduced on Monday: we work one example below, just to reinforce what we're doing here.

**Claim 2.2.**

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

*Proof.* First, examine the quantity

$$\frac{|x|}{x}.$$

For $x > 0$, we have that

$$\frac{|x|}{x} = 1;$$

therefore, for any $\epsilon > 0$, it doesn't even matter what $\delta$ we pick! – because for any $x$ with $0 < x$, we have that

$$\left| \frac{|x|}{x} - 1 \right| = 0 < \epsilon.$$

Thus, the limit as $\frac{|x|}{x}$ approaches 0 from the right hand side is 1, as claimed. □

One-sided limits are particularly useful when we're discussing limits at infinity, as we describe in the next section:

## 3. Limits at Infinity

**Definition 3.1.** For a function $f : X \to Y$, we say that

$$\lim_{x \to +\infty} f(x) = L$$

if and only if

(1) (vague:) as $x$ goes to "infinity," $f(x)$ goes to $L$.
(2) (concrete, symbols:)

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x > N) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x \to -\infty} f(x) = L$$

if and only if

(1) (vague:) as $x$ goes to "negative infinity," $f(x)$ goes to $L$.
(2) (concrete, symbols:)

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x < N) \Rightarrow (|f(x) - L| < \epsilon).$$

In class, we described a rather useful trick for calculating limits at infinity:

**Proposition 3.2.** *For any function* $f : X \to Y$*,*

$$\lim_{x \to +\infty} f(x) = \lim_{x \to 0^+} f\left(\frac{1}{x}\right).$$

*Similarly,*

$$\lim_{x \to -\infty} f(x) = \lim_{x \to 0^-} f\left(\frac{1}{x}\right).$$

The use of this theorem is that it translates limits at infinity (which can be somewhat complex to examine) into limits at 0, which can be in some sense a lot easier to deal with: as opposed to worrying about what a function does at extremely large values, we can just consider what a different function does at rather small values (which can make our lives often a lot easier.)

Here's an example, to illustrate where this comes in handy:

**Claim 3.3.**

$$\lim_{x \to +\infty} \frac{3x^2 + \cos(34x) + 10^7 \cdot x}{2x^2 + 1} = \frac{3}{2}.$$

*Proof.* Motivated by our proposition above, let us subsitute $1/x$ for $x$, so that we have

$$\lim_{x \to +\infty} \frac{3x^2 + \cos(34x) + 10^7 \cdot x}{2x^2 + 1} = \lim_{x \to 0^+} \frac{3(1/x)^2 + \cos(34/x) + 10^7 \cdot (1/x)}{2(1/x)^2 + 1}.$$

Multiplying both top and bottom by $x^2$, this limit is equal to

$$\lim_{x \to 0^+} \frac{3 + x^2 \cos(34/x) + 10^7 \cdot x}{2 + x^2}.$$

Because limits play nicely with arithmetic, we know that the limit of this ratio is the ratio of the two limits $3 + x^2 \cos(34/x) + 10^7 \cdot x$ and $2 + x^2$, **if and only iff both limits exist.**

But that's simple to see: because $2 + x^2$ is a polynomial, it's continuous, and thus

$$\lim_{x \to 0^+} 2 + x^2 = 2 + 0^2 = 2.$$

As well, because

$$3 - x^2 + 1 -^7 \cdot x \leq 3 + x^2 \cos(34/x) + 10^7 \cdot x \leq 3 + x^2 + 1 -^7 \cdot x,$$

and both of those polynomials converge to 3 as $x \to 0^+$, the squeeze theorem tells us that

$$\lim_{x \to 0^+} 3 + x^2 \cos(34/x) + 10^7 \cdot x = 3$$

as well.

Thus, because both limits exist, we have that

$$\lim_{x \to 0^+} \frac{3 + x^2 \cos(34/x) + 10^7 \cdot x}{2 + x^2} = \frac{\lim_{x \to 0^+}(3 + x^2 \cos(34/x) + 10^7 \cdot x)}{\lim_{x \to 0^+}(2 + x^2)} = \frac{3}{2},$$

as claimed. □

One useful application of limits at infinity comes through studying the intermediate value theorem, which is the subject of our next section:

## 4. The Intermediate Value Theorem

**Theorem 4.1.** *If $f$ is a continuous function on $[a, b]$, then $f$ takes on every value between $f(a)$ and $f(b)$ at least once.*

Most uses of this theorem occur when we have a continuous function $f$ that takes on both positive and negative values on some interval; in this case, the intermediate value theorem tells us that this function must have a zero between each pair of sign changes. Basically, when you have a question that's asking you to find zeroes of a function, or to show that a function with prescribed endpoint behavior takes on some other values, the IVT is the way to go.

To illustrate this, consider the following example:

**Claim 4.2.** *If $p(x)$ is an odd-degree polynomial, it has a root in $\mathbb{R}$ – i.e. there is some $x \in \mathbb{R}$ such that $p(x) = 0$.*

*Proof.* Write

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n,$$

where $n$ is an odd natural number and $a_n > 0$. (The case where $a_n < 0$ is identical to the proof we're about to do if you flip all of the inequalities, so we omit it here by symmetry.)

Then, notice that

$$\begin{aligned}
\lim_{x\to+\infty} \frac{a_0 + \ldots + a_n x^n}{x^n} &= \lim_{x\to+\infty} \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots + \frac{a_{n-1}}{x} + a_n\right) \\
&= \lim_{x\to+\infty} \left(\frac{a_0}{x^n}\right) + \lim_{x\to+\infty} \left(\frac{a_1}{x^{n-1}}\right) + \ldots + \lim_{x\to+\infty} (a_n) \\
&= 0 + \ldots + 0 + a_n \\
&= a_n,
\end{aligned}$$

(where the second line is justified because all of the individual limits exist.)

As a result, we know that for large positive values of $x$, $\frac{a_0+\ldots+a_n x^n}{x^n}$ is as close to $a_n$ as we would like. Specifically, we know that for large values of $x$, we have that the distance between $\frac{a_0+\ldots+a_n x^n}{x^n}$ and $a_n$ is less than, say, $a_n/2$. As a consequence, we have specifically that $\frac{a_0+\ldots+a_n x^n}{x^n}$ is **positive**, for large positive values of $x$ – thus, for some large positive $x$, we have that

$$x^n \cdot \frac{a_0 + \ldots + a_n x^n}{x^n} = (\text{positive}) \cdot (\text{positive}) = (\text{positive}).$$

Similarly, because

$$\begin{aligned}
\lim_{x\to-\infty} \frac{a_0 + \ldots + a_n x^n}{x^n} &= \lim_{x\to-\infty} \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \ldots + \frac{a_{n-1}}{x} + a_n\right) \\
&= \lim_{x\to-\infty} \left(\frac{a_0}{x^n}\right) + \lim_{x\to-\infty} \left(\frac{a_1}{x^{n-1}}\right) + \ldots + \lim_{x\to-\infty} (a_n) \\
&= 0 + \ldots + 0 + a_n \\
&= a_n,
\end{aligned}$$

we also have that for large **negative** values of $x$, $\frac{a_0+\ldots+a_n x^n}{x^n}$ is as close to $a_n$ as we'd like, and thus that $\frac{a_0+\ldots+a_n x^n}{x^n}$ is **positive**, for large **negative** values of $x$.

Thus, for some large negative value of $x$, we have that

$$x^n \cdot \frac{a_0 + \ldots + a_n x^n}{x^n} = (\text{negative}) \cdot (\text{positive}) = (\text{negative}).$$

(Notice that the fact that $n$ was odd was used in the above calculation, to insure that $x$ negative implies that $x^n$ is negative.)

We have thus shown that our polynomial adopts at least one positive and one negative value: thus, by the intermediate value theorem, it must be 0 somewhere between these two values! Thus, our polynomial has a root, as claimed. □

## 5. Open, Closed, and Bounded Sets

Finally, we make something of a detour here, to quickly define open, closed, and bounded sets:

**Definition 5.1.** A set $X \subset \mathbb{R}$ is called **open** if for any $x \in X$, there is some neighborhood $\delta_x$ of $x$ such that the entire interval $(x - \delta_x, x + \delta_x)$ lies in $X$.

**Example 5.2.**

- The sets $\mathbb{R}$ and $\emptyset$ are both trivially open sets.
- Any open interval $(a, b)$ is an open set.
- The union[1] of arbitrarily many open sets is open.
- The intersection [2] of finitely many open sets is open.

**Definition 5.3.** A set $X \subset \mathbb{R}$ is called **closed** if its complement[3] is open.

**Example 5.4.**

- The sets $\mathbb{R}$ and $\emptyset$ are both trivially closed sets. Note that this means that some sets can be both open and closed!
- Any closed interval $[a, b]$ is an closed set.
- The intersection of arbitarily many closed sets is closed.
- The union of finitely many closed sets is closed.

**Definition 5.5.** A set $X \subset \mathbb{R}$ is bounded iff there is some value $M \in \mathbb{R}$ such that $-M \le x \le M$, for any $x \in X$.

We will work more closely with these definitions in future lectures: however, for now, it suffices to note the following useful theorem, which we'll use heavily in our discussion of the derivative:

**Theorem 5.6.** *(Extremal value theorem:) If $f : X \to Y$ is a continuous function, and $X$ is a closed and bounded subset $X$ of $\mathbb{R}$, then $f$ attains its minima and maxima. In other words, there are values $m, M \in X$ such that for any $x \in X$, $f(m) \le f(x) \le f(M)$.*

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[1]The union $X \cup Y$ of two sets $X, Y$ is the set $\{a : a \in X \text{ or } a \in Y, \text{ or both.}\}$

[2]The intersection $X \cap Y$ of two sets $X, Y$ is the set $\{a : a \in X \text{ and } a \in Y.\}$

[3]The complement $X^c$ of a set $X$ is the set $\{a : a \notin X\}$