MATH 8, SECTION 1, WEEK 3 - RECITATION NOTES

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Abstract. These are the notes from Friday, Oct. 15th's lecture. In this talk, we study absolute and conditional convergence, and talk a little bit about the mathematical constant e.

1. Random Question

Question 1.1. *Suppose that $\{n_k\}$ is the sequence consisting of all natural numbers that don't have a 9 anywhere in their digits. Does the corresponding series*

$$\sum_{k=1}^{\infty} \frac{1}{n_k}$$

converge?

2. Absolute and Conditional Convergence: Definitions and Theorems

For review's sake, we repeat the definitions of absolute and conditional convergence here:

Definition 2.1. A series $\sum_{n=1}^{\infty} a_n$ converges **absolutely** iff the series $\sum_{n=1}^{\infty} |a_n|$ converges; it converges **conditionally** iff the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 2.2. The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally. This follows from our work in earlier classes, where we proved that the series itself converged (by looking at partial sums;) conversely, the absolute values of this series is just the harmonic series, which we know to diverge.

As we saw in class last time, most of our theorems aren't set up to deal with series whose terms alternate in sign, or even that have terms that occasionally switch sign. As a result, we developed the following pair of theorems to help us manipulate series with positive and negative terms:

Theorem 2.3. *(Alternating Series Test / Leibniz's Theorem) If $\{a_n\}_{n=0}^{\infty}$ is a sequence of positive numbers that monotonically decreases to 0, the series*

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

(Note: I was unsure in class if the sequence needs to monotonically decrease to 0. It definitely does! To see why, consider the sequence

$$\{2, 1, 2/2, 1/2, 2/3, 1/3, 2/4, 1/4, 2/5, 1/5, \dots\}.$$

The alternating series associated to this sequence is just $2-1+2/2-1/2+2/3-1/3+\ldots = 1+1/2+1/3+1/4\ldots$, i.e. the harmonic series, which we know diverges. So, monotonically decreasing is highly necessary!)

Theorem 2.4. *(Absolute convergence implies convergence:) If the series $\sum_{n=1}^{\infty}|a_n|$ converges, so does the series $\sum_{n=1}^{\infty}a_n$.*

This theorem is basically one of our only tools that can deal with a series whose terms alternate unpredictably; i.e. if you have $\sin(n)$ terms running about, this theorem can be quite useful.

To illustrate the use of these theorems, we have two examples in the next section:

3. Absolute and Conditional Convergence: Examples

Question 3.1. *Determine whether the series*

$$\sum_{n=1}^{\infty}\frac{(-10)^n}{n^n}$$

converges, and whether that convergence is absolute or conditional.

Proof. To see that this series converges, simply apply the alternating series test. Specifically, because we have that

$$0 \leq \left(\frac{10}{n}\right)^n \leq \left(\frac{10}{11}\right)^n,$$

for all $n \geq 11$, the squeeze theorem tells us that that this sequence converges to 0; furthermore, we know that this convergence is monotone, as (for $n \geq 11$)

$$\left(\frac{10}{n}\right)^n < \left(\frac{10}{n}\right)^{n+1} < \left(\frac{10}{n+1}\right)^{n+1}.$$

Thus, the alternating series test tells us that the series $\sum_{n=1}^{\infty}\frac{(-10)^n}{n^n}$ converges.

For absolute convergence: simply apply the root test to the series $\sum_{n=1}^{\infty}\left|\frac{(-10)^n}{n^n}\right|$. Because

$$\lim_{n\to\infty}\frac{10^n}{n^n} = 0 < 1,$$

we know that this series must converge; so our original series must converge absolutely. ☐

Claim 3.2. *If the series $\sum_{n=1}^{\infty}a_n$ converges absolutely, then so does the series $\sum_{n=1}^{\infty}a_n\cdot\cos^3(a_n)$.*

Proof. So: we apply the comparison test. Because

$$-1 \leq \cos^3(a_n) \leq 1,$$

we know that

$$0 \leq \left|a_n\cos^3(a_n)\right| \leq |a_n|.$$

So, by the comparison test, because

$$\sum_{n=1}^{\infty}|a_n|$$

converges, so does

$$\sum_{n=1}^{\infty} \left| a_n \cos^3(a_n) \right| .$$

$\square$

4. Conditional Convergence and Rearranging Sums, Revisited

In class on Friday, Dr. Ramakrishnan made the following claim: if $\sum_{n=1}^{\infty} a_n$ was a conditionally convergent series, then for any real number r, there is some way to rearrange $\sum_{n=1}^{\infty} a_n$'s terms so that they sum to r.

Let's prove that.

First, we need the following useful pair of lemmas:

Lemma 4.1. *Given a sequence $\{a_n\}$, define the sequences$\{a_n^+\}$ and $\{a_n^-\}$ as follows:*

- *$a_n^+ = a_n$ if $a_n > 0$, and 0 otherwise.*
- *$a_n^- = a_n$ if $a_n < 0$, and 0 otherwise.*

We claim that if the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, both of the sums $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ must diverge.

Proof. Start by making the following observations about the $a_n^{\pm}$;

- By definition, the a_n^+'s are just the positive a_n's and the a_n^-'s are just the negative a_n's: so we have that
$$\begin{aligned} & a_n = a_n^+ + a_n^-, \forall n \\ \Rightarrow \qquad & |a_n| = |a_n^+ + a_n^-| = a_n^+ - a_n^-, \forall n. \end{aligned}$$
- As well, because of the above identity, we have that
$$\begin{aligned} & |a_n| = a_n^+ - a_n^-, a_n = a_n^+ + a_n^-, \forall n \\ \Rightarrow \qquad & a_n^+ = \frac{a_n + |a_n|}{2}, a_n^- = \frac{a_n - |a_n|}{2}, \forall n. \end{aligned}$$

Applying these identities to our sums in question tells us that

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n| &= \left(\sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^- \right), \\ \Rightarrow \sum_{n=1}^{\infty} a_n^+ &= \frac{1}{2} \left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} |a_n| \right), \\ \Rightarrow \sum_{n=1}^{\infty} a_n^- &= \frac{1}{2} \left(\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} |a_n| \right). \end{aligned}$$

Thus, if either of the sums $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ converged, we would have that the sum $\sum_{n=1}^{\infty} |a_n|$ converges, by adding the convergent series $2\sum_{n=1}^{\infty} a_n$ to whichever of the signed series converged. As we know that $\sum_{n=1}^{\infty} |a_n|$ diverges, this must be impossible; so both $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ must diverge. $\square$

Lemma 4.2. *(Vanishing criterion) If $\{a_n\}$ is summable, then $\lim_{n\to\infty} a_n = 0$.*

(we've proven this in class before, so I omit the proof here; if you've forgotten how it goes, try applying the Cauchy criterion.)

So, given our two lemmas, we move on to our main result:

Theorem 4.3. *Pick any series $\sum_{n=1}^{\infty} a_n$ that converges conditionally, and choose any $r \in \mathbb{R}$. Then there is a rearrangement of the sequence$\{a_n\}_{n=1}^{\infty}$ into a sequence $\{b_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} b_n = r$.*

(In other words: if $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent sequence, then by switching its terms around we can make it sum up to any number we want.)

Proof. Take any conditionally convergent series $\sum_{n=1}^{\infty} a_n$, and suppose without any loss of generality that the constant r we want to rearrange $\sum_{n=1}^{\infty} a_n$ to sum to is positive. (The proof will look almost identical in the case $r \leq 0$, so we will omit that case here.)

Given our series $\sum_{n=1}^{\infty} a_n$, define the following two sequences $\{p_n\}_{n=1}^{\infty}, \{q_n\}_{n=1}^{\infty}$ as follows:

$$\begin{aligned} \{p_n\} &:= \text{ the positive terms of the } a_n\text{'s} \\ \{q_n\} &:= \text{ the negative terms of the } a_n\text{'s.} \end{aligned}$$

For example, if $\{a_n\}_{n=1}^{\infty}$ was the sequence $\left\{\frac{(-1)^{n+1}}{n}\right\}$, then

$$\begin{aligned} \{p_n\} &:= \left\{\frac{1}{2n-1}\right\}_{n=1}^{\infty}, \text{ and} \\ \{q_n\} &:= \left\{-\frac{1}{2n}\right\}_{n=1}^{\infty}. \end{aligned}$$

By our earlier lemma, we then know that both of the sums $\sum_{n=1}^{\infty} p_n$ or $\sum_{n=1}^{\infty} q_n$ must diverge. So: because $\sum_{n=1}^{\infty} p_n$ diverges, we know that there is some M_1^+ such that $\sum_{n=1}^{M_1^+} p_n > r$. Pick the smallest such M_1^+ such that this is true: i.e pick M_1^+ such that

$$\sum_{n=1}^{M_1^+-1} p_n \leq r, \sum_{n=1}^{M_1^+} p_n > r.$$

Note further that, by the construction above, that M_1^+ in some sense marks the "best approximation" to r that we can get by adding up the values of the a_n's; i.e. that

$$\left|\sum_{n=1}^{M_1^+} p_n - r\right| < p_{M_1^+}.$$

Then, because $\sum_{n=1}^{\infty} q_n$ also diverges, we know that there is some value M_1^- such that $\sum_{n=1}^{M_1^+} p_n + \sum_{n=1}^{M_1^-} q_n < r$; again, pick the smallest value such that this holds. This, again, puts us in the situation such that

$$\sum_{n=1}^{M_1^+} p_n + \sum_{n=1}^{M_1^- -1} q_n \geq r, \sum_{n=1}^{M_1^+} p_n + \sum_{n=1}^{M_1^-} q_n < r.$$

Again, because of our construction above, we can think of M_1^- as another "best approximation" to r, in that

$$\left| \sum_{n=1}^{M_1^+} p_n + \sum_{n=1}^{M_1^-} q_n - r \right| < |q_{M_1^-}|.$$

Repeat this process! In other words, keep picking values M_k^+, M_k^- such that the sums

$$\sum_{n=1}^{M_1^+} p_n + \sum_{n=1}^{M_1^-} q_n + \ldots | \sum_{n=1}^{M_k^+} p_n + \sum_{n=1}^{M_k^-} q_n$$

are increasingly good approximations of r: i.e. that

$$\left| \sum_{n=1}^{M_1^+} p_n + \sum_{n=1}^{M_1^-} q_n + \ldots | \sum_{n=1}^{M_k^+} p_n + \sum_{n=1}^{M_k^-} q_n - r \right| < |q_{M_k^-}|.$$

Take this summation as our rearrangement! By construction, we have that the distance of the partial sums from r isl bounded by the $|q_{M_k^-}|$'s, which go to 0 as $k \to \infty$; so we know that these partial sums converge to r. But this means that our rearranged series converges to r! So we're done. □

5. Defining e

So: in Math 8 on Wednesday, we defined e as follows:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

In Math 1a on Friday, we defined e via a power series as follows:

$$e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!}.$$

Which of these is right? Well, as it turns out: both! I.e. both of these objects are equal. To prove this, we need just one simple lemma, and a few calculations:

Lemma 5.1. *For any real numbers x, y, we have*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

where

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

(If you haven't proved this before: try induction! It's relatively simple and straightforward.)

Claim 5.2. *For any $x \in \mathbb{R}$, we have*

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=1}^{\infty}\frac{x^k}{k!}.$$

Proof. Using our lemma, write

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k}\\ &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}\cdot\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}.\end{aligned}$$

Because $\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k} \leq \frac{n^k}{n^k} = 1$, we have the simple upper bound

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}\cdot\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}\\ &\leq \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}\\ &= \sum_{k=1}^{\infty}\frac{x^k}{k!}.\end{aligned}$$

Thus, it suffices to show that $\sum_{k=1}^{\infty}\frac{x^k}{k!}$ is also a lower bound on $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

To see this: notice that

$$\frac{(n-k+1)^k}{n^k} \leq \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k} \leq \frac{n^k}{n^k}.$$

Thus, because for any fixed k the left and right hand sides above go to 1 as $n\to\infty$, the squeeze theorem tells us that

$$\lim_{n\ to\infty}\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k} = 1,$$

for any fixed k. Thus, by the definition of convergence, we know that for any fixed k and $\epsilon > 0$, there is a N such that for all $n > N$,

$$1 > \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k} > 1-\epsilon.$$

Also, because

$$\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k} < \frac{n\cdot(n-1)\cdot\ldots\cdot(n-(k-1)+1)}{n^{k-1}},$$

we know in fact that for any fixed K and $\epsilon > 0$, there's a N such that for all $n > N$ and $k \leq K$, we have

$$1 > \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k} > 1-\epsilon.$$

But this tells us that

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}\cdot\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}. \\
&\geq \lim_{n\to\infty}\sum_{k=0}^{K}\frac{x^k}{k!}\cdot\frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}. \\
&\geq \sum_{k=0}^{K}\frac{x^k}{k!}\cdot(1-\epsilon).
\end{aligned}$$

Letting $K\to\infty$ and $\epsilon\to 0$ gives us that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \geq \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

Thus, we've proven that these two definitions of e^x are equivalent. $\square$