MATH 8, SECTION 1, WEEK 2 - RECITATION NOTES

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Abstract. These are the notes from Wednesday, Oct. 6th's lecture, where we showed that there are infinite sets with different cardinalities.

1. Random Question

Question 1.1. *Define $\alpha(n)$ as the function that takes in an integer and returns the total number of prime divisors of n: for example $\alpha(9) = 2, \alpha(36) = 4$, and $\alpha(7) = 1$. Find*

$$\lim_{n\to\infty} \frac{\alpha(n)}{n}.$$

2. Introduction

The goal of this lecture is to prove the following:

Theorem 2.1. *The sets $\mathbb{N}$ and $\mathbb{R}$ have different cardinalities.*

In our last lecture, we introduced the idea of cardinality, which allows us to make sense of the idea of having "different" sizes of infinity; the last tool we need here before we begin our proof, then, is to define the sets we are working with. The natural numbers are (hopefully, by this point) something we understand quite well: the reals, however, we have not yet given a rigorous footing.

Let's change that.

3. Defining the Real Numbers

In $\mathbb{Q}$, there are sequences $\{a_n\}_{n=1}^{\infty}$ of rational numbers that have the three following properties:

- the terms a_n are increasing,
- there is an upper bound for the terms a_n, but
- this sequence of a_n's doesn't converge to a rational number.

Specifically, consider the following example:

- $a_0 = 1$
- $a_1 = 1.4$
- $a_2 = 1.41$
- $a_n =$ the first n decimal places of $\sqrt{2}$; i.e.

$$a_n = \frac{\lfloor \sqrt{2} \cdot 10^n \rfloor}{10^n}$$

Proposition 3.1. *The sequence $\{a_n\}_{n=1}^{\infty}$ defined above converges to $\sqrt{2}$, an irrational number.*

Proof. Before we begin our proof, we should probably say just what convergence *is:*

Definition 3.2. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value λ if, for any distance ϵ, the a_n's are eventually within ϵ of λ. To put it more formally, $\lim_{n\to\infty} a_n = \lambda$ iff for any distance ϵ, there is some cutoff point N such that for any n greater than this cutoff point, a_n must be within ϵ of our limit λ.

To put it formally with a bunch of arcane symbols:

$$\lim_{n\to\infty} a_n = \lambda \text{ iff } (\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

Whenever you can, think of the words and not the symbols; it's often a lot easier to get confused by a string of quantifiers than it is to be confused by plain English.

Thus, if we want to show

$$\lim_{n\to\infty} a_n = \sqrt{2},$$

we just need to show that for any $\epsilon > 0$ we can find a cutoff point N past which all of the a_n's are within ϵ of $\sqrt{2}$. So: let's examine the a_n's!

Specifically, look at the quantity

$$|a_n - \sqrt{2}|.$$

By definition, because a_n is just the first n decimal places of $\sqrt{2}$, this is the number that's 0 for its first n decimal places and then corresponds to the rest of the digits of $\sqrt{2}$. But such a number is rather small – in fact, it's always less than 10^{-n}, as its first n decimal places are 0.

Thus , we have that

$$|a_n - \sqrt{2}| < 10^{-n}, \text{ for any } n.$$

With this noticed, it's clear what our cutoff point should be – if we want to have $|a_n - \sqrt{2}| < \epsilon$, it suffices to force $10^{-n} < \epsilon$. So pick N such that this happens: i.e. pick N such that

$$10^{-N} < \epsilon.$$

Then, we have that for any $n > N$,

$$|a_n - \sqrt{2}| < 10^{-n} < 10^{-N} < \epsilon.$$

But this is the definition of convergence! In other words, for any $\epsilon > 0$ we've found a cutoff point N past which all of the a_n's are within ϵ of $\sqrt{2}$. So we've proven that $\sqrt{2}$ is indeed the limit, and we are done! □

It bears noting that normal proofs of convergence are much shorter than the above: here, we have taken care to show every detail clearly, so that it's obvious what we've done and why we've done it. Future proofs will be much less pedantic!

So: there are sequences of rational numbers where

- the terms a_n are increasing,
- there is an upper bound for the terms a_n, but
- this sequence of a_n's doesn't converge to a rational number.

Can these things exist for real numbers? as it turns out, no!

Lemma 3.3. *If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that*

- *the terms a_n are increasing, and*

- *there is an upper bound for the terms* a_n*,*

then the sequence converges to some real number.

Proof. So: let A be the set consisting of all of the numbers a_n in our sequence. A is bounded above, because $\{a_n\}_{n=1}^{\infty}$ is; so A has a **least upper bound**! Call it α. So: we claim that

$$\lim_{n\to\infty} a_n = \alpha.$$

To see this: choose any $\epsilon > 0$. Because α is a least upper bound, we know that no number smaller than α can be an upper bound – so, in specific, $\alpha - \epsilon$ is not an upper bound. But what does this mean? Just that there is some $a_N \in A$ such that

$$a_n > \alpha - \epsilon \Leftrightarrow \epsilon > \alpha - a_N.$$

But because the a_n are nondecreasing and bounded above by α, we know that actually

$$\epsilon > |\alpha - a_n|, \forall n > N.$$

So $\lim_{n\to\infty} a_n$ converges (specifically, to α,) just as we claimed. □

This is a kind of cool property of the real numbers: one that, in fact, we can use to define the real numbers! Specifically: take any real number r, and let $\{a_n\}^{|}infty_{n=0}$ be sequence given by taking a_n to be r approximated to its first n decimal places: i.e.

$$a_n = \frac{\lfloor r \cdot 10^n \rfloor}{10^n}.$$

Then, by a proof exactly the same as before, this sequence converges to r. Motivated by this, we can use this to actually **define** the real numbers as anything you can get by such a decimal expansion! I.e. we have the following:

Definition 3.4. The real numbers $\mathbb{R}$ are just the set of all possible limits of increasing sequences of rational numbers: explicitly, to put it in a nice form,

$$\mathbb{R} = \{n + .a_0a_1a_2 \ldots : n \in \mathbb{Z}, a_i \in [0...9]\}$$

where we regard things like .9999999 . . . to be equal to 1.00000 . . .

4. Examining $\mathbb{R}$ and $\mathbb{N}$

With this definition at hand, we can now move on to our main goal:

Theorem 4.1. $\mathbb{R}$ *is of a strictly greater cardinality than* $\mathbb{N}$*.*

Specifically, we will prove the following stronger claim:

Proposition 4.2. *The set* $[0, 1]$ *is of a strictly greater cardinality than* $\mathbb{N}$*.*

Proof. (This is Cantor's famous diagonalization argument.) Suppose not – that they were the same cardinalities. As a result, there is a bijection between these two sets! Pick such a bijection $f : \mathbb{N} \to [0, 1]$.

For every $n \in \mathbb{N}$, look at the number $f(n) \in [0, 1)$. This is a number between 0 and 1; so it has a decimal representation. Pick numbers $a_{n,1}, a_{n,2}, a_{n,3}, \ldots$ that correspond to this decimal representation – i.e. pick numbers $a_{n,i}$ such that

$$f(n) = .a_{n,1}a_{n,2}a_{n,3} \ldots$$

For example, if $f(4) = .125$, we would pick $a_{4,1} = 1, a_{4,2} = 2, a_{4,3} = 5$, and $0 = a_{4,4} = a_{4,5} = a_{4,6} = \ldots$, because the first three digits are 1,2, and 5, and the rest of them are zeroes.

Now, write these numbers in a table, as below:

$f(1)$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$\ldots$
$f(2)$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	
$f(3)$	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	
$f(4)$	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$a_{4,4}$	
$\vdots$	$\vdots$				$\ddots$

In particular, look at the entries $a_{1,1}a_{2,2}a_{3,3}\ldots$ on the diagonal. We define a number B using these digits as follows:

- Define $b_i = 2$ if $a_{i,i} \neq 2$, and $b_i = 8$ if $a_{i,i} = 2$.
- Define B to the be the number with digits given by the b_i – i.e.

$$B = .b_1b_2b_3b_4\ldots$$

From our discussion before, this generates a real number B that's just a string of 2's and 8's, andthus that lies in $[0, 1)$. So, if our function f is indeed a bijection, it must be in particular a surjection: consequently, there must be some k such that $f(k) = B$. But the k-th digit of $f(k)$ is $a_{k,k}$ by construction, and the k-th digit of B is b_k – by construction, these are different numbers! So $f(k) \neq B$: a contradiction.

Thus, no such bijection can exist: as a result, one of these two sets is of a strictly greater cardinality than the other. To see which is greater: notice that there is in fact an injection of the natural numbers into $[0, 1]$, given by $f(n) = 1/n$. Consequently, the natural numbers are the "smaller" of the two sets, as $[0, 1]$ contains a subset – $\{1/n : n \in \mathbb{N}\}$ – that is the same size as $\mathbb{N}$. □