MATH 8, SECTION 1, WEEK 2 - RECITATION NOTES

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Abstract. These are the notes from Monday, Oct. 4th's lecture, where we discussed whether we could define a meaningful concept of "size" for infinite sets.

1. Random Question

This is slightly different from the version I posted in class: there, I forgot that we should allow polynomials with an infinite number of positive-indexed terms! Oops. The corrected version is posted below:

Question 1.1. *Let*

$$E = \left\{ \sum_{i=-\infty}^{\infty} a_i x^i : a_i \in \mathbb{R} \right\};$$

in other words, E consists of all of the "generalized" polynomials over $\mathbb{R}$ (polynomials where you can have both positive and negative coefficients, and possibly infinitely many of either.)

Let F be the subset of E made of those polynomials with only finitely many negative-power terms: in other words, let

$$F = \{p(x) \in E : p(x) \text{ has only finitely many nonzero coefficients attached to negative powers of } x.\}$$

Define addition and multiplication as normal for elements in F: i.e.

$$\left(\sum_{i=-\infty}^{\infty} a_i x^i \right) + \left(\sum_{i=-\infty}^{\infty} b_i x^i \right) = \sum_{i=-\infty}^{\infty} (a_i + b_i) x^i,$$

$$\left(\sum_{i=-\infty}^{\infty} a_i x^i \right) \cdot \left(\sum_{i=-\infty}^{\infty} b_i x^i \right) = \sum_{i=-\infty}^{\infty} \left(\sum^{|} infty_{j=-\infty} a_j b_{i-j} \right) x^i.$$

Define an order relation on F by saying

$$p(x) = \sum_{i=-\infty}^{\infty} a_i x^i > 0 \text{ iff } a_i > 0, \text{ where } i \text{ is the smallest integer such that } a_i \text{ is a nonzero coefficient of } p(x).$$

Show that this is an ordered field that contains $\mathbb{N}$, in which $\mathbb{N}$ is bounded.

2. Sizes of Infinity: Introduction

What does it mean for two sets to be the same size? In the finite case, this question is rather trivial; for example, we know that the two sets

$$A = \{1, 2, 3\}, \qquad\qquad B = \{A, B, \text{emu}\}$$

are the same size because they both have the same *number* of elements – in this case, 3.

But what about infinite sets? For example, look at the sets

$$\mathbb{N}, \qquad \mathbb{Q}, \qquad \mathbb{R}, \qquad \mathbb{C};$$

are any of these sets the same size? Are any of them larger? By how much?

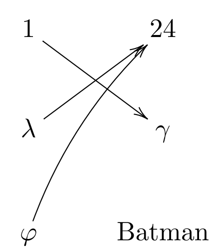
In the infinite case, the tools we used for the finite – counting up all of the elements – don't work. In response to this, we are motivated to try to find another way to count: in this case, one that involves **functions**.

3. Functions (formally defined)

Definition 3.1. A **function** f with domain A and range B, formally speaking, is a collection of pairs (a, b), with $a \in A$ and $b \in B$, such that there is exactly one pair (a, b) for every $a \in A$. More informally, a function $f : A \to B$ is just a map which takes each element in A to some element of B.

Examples 3.2.

- $f : \mathbb{Z} \to \mathbb{N}$ given by $f(n) = 2|n| + 1$ is a function.
- $g : \mathbb{N} \to \mathbb{N}$ given by $f(n) = 2|n| + 1$ is a function; in fact, it is a different function, because it has a different domain!
- The function h depicted below by the three arrows is a function, with domain $\{1, \lambda, \varphi\}$ and range $\{24, \gamma, \text{ Batman}\}$:



This may seem like a silly example, but it's illustrative of one key concept: functions are just **maps between sets!** Often, people fall into the trap of assuming that functions have to have some nice "closed form" like $x^3 - \sin(x)$ or something, but that's not true! Often, functions are either defined piecewise, or have special cases, or are generally fairly ugly/awful things; in these cases, the best way to think of them is just as a collection of arrows from one set to another, like we just did above.

Now that we've formally defined functions and have a grasp on them, let's introduce a pair of definitions that will help us with our question of "size:"

Definition 3.3. We call a function f **injective** if it never hits the same point twice – i.e. for every $b \in B$, there is **at most one** $a \in A$ such that $f(a) = b$.

Example 3.4. The function h from before is not injective, as it sends both λ and φ to 24:

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[uur, bend left] & \text{Batman}
\end{tikzcd}$$

However, if we add a new element π to our range, and make φ map to π, our function is now injective, as no two elements in the domain are sent to the same place:

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[dr, bend right] & \text{Batman} \\
 & \pi
\end{tikzcd}$$

One observation we can quickly make about injective functions is the following:

Proposition 3.5. *If $f : A \to B$ is an injective function and A, B are finite sets , then size(A) $\leq$size(B). (Formally, we write $|A| \leq |B|$, and use the vertical brackets around a set to denote its size.)*

The reasoning for this, in the finite case, is relatively simple:

(1) If f is injective, then each element in A is being sent to a different element in B.
(2) Thus, you'll need B to have at least $|A|$-many elements to provide that many targets.

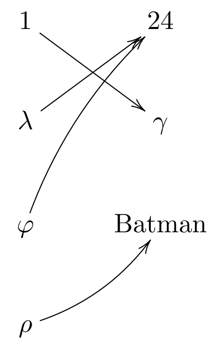
A converse concept to the idea of injectivity is that of **surjectivity**, as defined below:

Definition 3.6. We call a function f **surjective** if it hits every single point in its range – i.e. if for every $b \in B$, there is **at least one** $a \in A$ such that $f(a) = b$.

Example 3.7. The function h from before is not injective, as it doesn't send anything to Batman:

$$\begin{tikzcd}
1 \arrow[dr] & 24 \\
\lambda \arrow[ur] & \gamma \\
\varphi \arrow[uur, bend left] & \text{Batman}
\end{tikzcd}$$

However, if we add a new element ρ to our domain, and make ρ map to Batman, our function is now surjective, as it hits all of the elements in its range:



As we did earlier, we can make one quick observation about what surjective functions imply about the size of their domains and ranges:

Proposition 3.8. *If $f : A \to B$ is an surjective function and A, B are finite sets , then $|A| \geq |B|$.*

Basically, this holds true because

1. Thinking about f as a collection of arrows from A to B, it has precisely $|A|$-many arrows by definition, as each element in A gets to go to precisely one place in B.
2. Thus, if we have to hit every element in B, and we start with only $|A|$-many arrows, we need to have $|A| \geq |B|$ in order to hit everything.

So: in the finite case, if $f : A \to B$ is injective, it means that $|A| \leq |B|$, and if f is surjective, it means that $|A| \geq |B|$. This motivates the following definition and observation:

Definition 3.9. We call a function **bijective** if it is both injective and surjective.

Proposition 3.10. *If $f : A \to B$ is an bijective function and A, B are finite sets , then $|A| = |B|$.*

Unlike our earlier idea of counting, this process of "finding a bijection" seems like something we can do with any sets – not just finite ones! As a consequence, we are motivated to make this our **definition** of size! In other words, we have the following definition:

Definition 3.11. We say that two sets A, B are the same size (formally, we say that they are of the same **cardinality**,) and write $|A| = |B|$, if and only if there is a bijection $f : A \to B$.

4. Sizes of Infinity: The Natural Numbers

Armed with a definition of size that can actually deal with infinite sets, let's start with some calculations to build our intuition:

Question 4.1. *Are the sets $\mathbb{N}$ and $\mathbb{N} \cup \{lemur\}$ the same size?*

Answer. Well: we know that they can be the same size iff there is a bijection between one and the other. So: let's try to make a bijection! In the typed notes, the suspense is somewhat gone, but (at home) imagine yourself taking a piece of paper, and writing out the first few elements of $\mathbb{N}$ on one side and of $\mathbb{N} \cup \{\text{lemur}\}$ on the other side. After some experimentation, you might eventually find yourself with the following map:

$$\begin{array}{ccc} \boxed{\mathbb{N}} & & \boxed{\mathbb{N} \cup \{\text{lemur}\}} \\ 1 & \longrightarrow & \text{lemur} \\ 2 & \longrightarrow & 1 \\ 3 & \longrightarrow & 2 \\ 4 & \longrightarrow & 3 \\ 5 & \longrightarrow & 4 \\ 6 & \longrightarrow & 5 \\ \vdots & & \vdots \end{array}$$

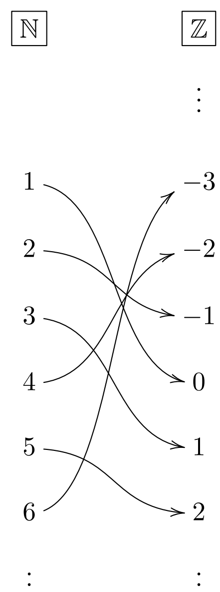
i.e. the map which sends 1 to the lemur and sends $n \to n-1$, for all $n \geq 2$. This is clearly a bijection; so these sets are the same size!

In a rather crude way, we have shown that adding one more element to a set as "infinitely large" as the natural numbers doesn't do anything to it! – the extra element just gets lost amongst all of the others.

This trick worked for one additional element. Can it work for infinitely many? Consider the next proposition:

Proposition 4.2. *The sets $\mathbb{N}$ and $\mathbb{Z}$ are the same cardinality.*

Proof. Consider the following map:



i.e. the map which sends $n \to (n-1)/2$ if n is odd, and $n \to -n/2$ if n is even. This, again, is clearly a bijection; so these sets are the same cardinality. $\square$

So: we can in some sense "double" infinity! Crazy – yet also sensical, after a fashion. After all, don't the natural numbers contain two copies of themselves in the even and odd numbers? And isn't that observation just what we used to turn $\mathbb{N}$ into $\mathbb{Z}$?

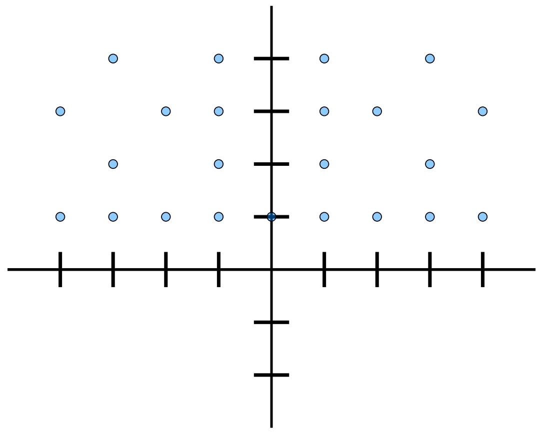
After these last two results, you might be beginning to feel like all of our infinite sets are the same size. In that case, the next result will hardly surprise you:

Proposition 4.3. *The sets $\mathbb{N}$ and $\mathbb{Q}$ are the same cardinality.*

Proof. First, recall how we've defined the rational numbers:

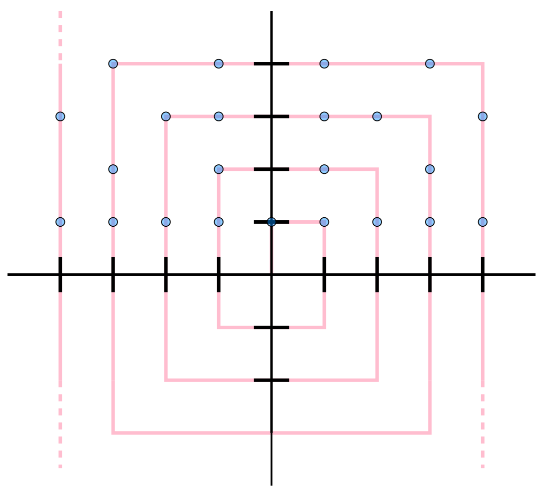
$$\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}, GCD(p,q) = 1\}.$$

Given this definition, we can think of these rational numbers as just points (p,q) in the integer plane $\mathbb{Z} \times \mathbb{Z}$:



Each rational number p/q with $GCD(p,q) = 1$, $p > 0$
represents a unique blue dot in the picture above.

On this picture, draw the spiral that starts at $(0,0)$ and goes through every point of $\mathbb{Z} \times \mathbb{Z}$, depicted below:



Let $f : \mathbb{N} \to \mathbb{Q}$ be defined by setting $f(n)$ to be the n-th rational point found by starting at (0,0) and walking along the depicted spiral pattern. This function hits every rational number exactly once by contstruction; thus, it is a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Consequently, $\mathbb{N}$ and $\mathbb{Q}$ are the same cardinality.

$\square$