MATH 8, SECTION 1, WEEK 1 - RECITATION NOTES

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Abstract. These are the notes from Monday, Sept. 27th's lecture. Here, we introduce some basic logical notation, and begin to discuss what it means to "prove" something.

1. Administrivia and Announcements

- Email: padraic@caltech.edu
- Website: www.its.caltech.edu/∼padraic Course notes for every lecture are posted here, ideally within a day of the lecture.
- Office: 360 Sloan
- Office Hours: MWF, 3-4 pm, in Sloan 151 / by request!
- Grading Policy: Math 8 is a 3-credit class that meets for three hours a week. Accordingly, its grading is strictly attendance-based! Registered students are required to attend at least 3/4 of the lectures to pass the course – effectively, this means you're allowed to miss at most 7 classes over the quarter. Absences are excused only if they are accompanied by a note from the Deans or from the Student Health Center; make-up work for missed classes is not available.
- Unregistered Students: Unregistered students are more than welcome to attend as many lectures as they want, and read the course notes online!

2. Random Question

Every class (that I remember to do this for,) I'll put up a question or two for people to think about during lecture; that way, if you've seen some of the material we're covering before and want something else to ponder, you won't be bored. These are random, mostly mathematical puzzles I've ran into in my career as a mathematician that I liked – if you're interested in any of them or happen to solve one of them, talk to me! I'm always happy to hear possible solutions or offer hints. Alternately, if you're not interested, don't worry; these are just for the curious/easily-distracted among you.

Question 2.1. *Can you find 4 points in the plane so that the distance between any two of them is odd?*

3. What is Math 8?

Math 8 is an auxiliary course for Math 1a; where Math 1a is (by design!) a highly abstract course, Math 8 is a much more hands-on and example-oriented series of lectures. Specifically, Math 8's lectures are designed to serve as a "how-to" guide on how to craft clear, well-written mathematical proofs; students who are new to the concept of formal proofs or who are otherwise concerned with meeting the levels of rigor in Math 1a are encouraged to attend!

Throughout Math 8, we're going to try to develop four main themes:

(1) The **language** of proofs. Throughout the course, we will continually introduce and define mathematical symbols and shorthand, so that you can read and write mathematics fluently.
(2) The **methods** of proof. There are many different styles of proof used throughout mathematics; in this class, we will focus on developing four specific styles (construction, deduction, induction, contradiction), discuss where to use each of these styles, and possibly mention some other more esoteric methods (probabilistic methods, anyone?)
(3) The **art** of proof. Why do we prove things? What constitutes a proof? How do you craft clear, concise proofs? We will talk about how to do all of these things continually through our course.
(4) Proofs in Calculus. Once we've developed some vocabulary and sophistication, we will begin to discuss how to specifically prove things and work in Calculus; topics like $\epsilon - \delta$ proofs, how to manipulate sequences/series/sums, and how to "think" in Calculus will be developed.

So: without any further ado, let's start!

4. Basic Vocabulary: Propositional Logic

Definition 4.1. A **proposition** is merely a statement that is either true or false (but not neither, or both!)

Examples 4.2. The statement "I am a lemur" is a proposition, as are the statements "The cake is a lie" and "Socrates is a man."

Take some proposition, and denote it by the letter P for short. What can we do with it?

- **Negation:** Given a proposition P, we can form the proposition not-P, denoted $\neg P$. $\neg P$ is the proposition defined to be true if and only if P is false.

  Example 4.3. The negation of "I am a lemur" is "I am not a lemur."

- **And:** Given a pair of propositions P and Q, we can form the proposition P-and-Q, denoted $P \wedge Q$. $P \wedge Q$ is the proposition defined to be true if and only if both P and Q are true, and false otherwise.
- **Or:** Given a pair of propositions P and Q, we can form the proposition P-or-Q, denoted $P \vee Q$. $P \vee Q$ is the proposition defined to be true if and only if either P is true, Q is true, or both P and Q are true; it is only false when both P and Q are false.
- **Implies:** Given a pair of propositions P and Q, we can form the proposition P-implies-Q, denoted $P \Rightarrow Q$. $P \Rightarrow Q$ is the proposition defined to be true if and only if P implies Q – in other words, it is true if either P is false and Q is anything, or P is true and Q is true. The only case where it is false is when P is true and Q is false.
- **Equivalent:** Given a pair of propositions P and Q, we can form the proposition P-iff-Q, denoted $P \Leftrightarrow Q$. $P \Leftrightarrow Q$ is the proposition defined to be

true if and only if both P and Q agree; i.e. if P and Q are both true, or if P and Q are both false. It is false whenever P and Q disagree.

A quick truth table, to summarize:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

So we have some vocabulary! Here are a few (optional) exercises to test these concepts:

(1) If $P \Rightarrow Q$ and $Q \Rightarrow P$, does that mean that $P \Leftrightarrow Q$? Prove it!
(2) What is the negation of the sentence "If I am the queen of France, then I am twelve feet tall" ? In general, what is a simpler way of formally writing the sentence "$\neg(P \Rightarrow Q)$"?
(3) Prove de Morgan's laws: that $\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$, and $\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$.

5. What is a Proof?

But what does it mean to even "prove" anything? A completely reasonable question you might have, after reading the above questions; after all, the answers to some of them are just obvious, right? What would it mean to prove such things?

Well: in mathematics, a **proof** is a deductive argument that, from some collection of previously established truths, shows that some given statement – the thing that we seek to prove – is true. Essentially, writing a proof is like building a tower out of blocks; you start out with a collection of smaller established objects and "stack" them up to get a new object. A few examples will help to illustrate what we're talking about:

Lemma 5.1. *Suppose that P and Q are propositions. Then $(P \Rightarrow Q) \leftrightarrow (\neg Q \Rightarrow \neg P)$ is true.*

Proof. Well: what does the above statement mean? It means the proposition on the left hand side – $(P \Rightarrow Q)$ – is true if and only if the proposition on the right hand side – $(\neg Q \Rightarrow \neg P)$ – is true. So, to prove that this proposition holds, we just need to examine when the left and right hand sides are true, and show that these cases coincide.

So: by definition, $(P \Rightarrow Q)$ is true if and only if $(P = true, Q = false)$ does not happen. Similarly, by definition, $(\neg Q \Rightarrow \neg P)$ is true if and only if $(\neg Q = true, \neg P = false)$ doesn't happen; but (by the definition of $\neg$) that only happens when $(Q = false, P = true)$. So the left hand side is indeed true only when the right hand side is! Thus, by the definition of $\Leftrightarrow$, we have proven our proposition to be true. □

Lemma 5.2. *The product of any two odd numbers is odd.*

Proof. Pick any two odd numbers x and y. We seek to show that xy is also odd.

So: where do we start? From the (very few) things we already know:

(1) the definition of an odd number: i.e. because x is odd, there must be an integer a such that $x = 2a+1$. Similarly, because y is odd, there is another integer b such that $y = 2b+1$.
(2) the axioms of arithmetic: specifically, we know that

$$xy = (2a+1)(2b+1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$$

by applying the distributive axiom twice. Furthermore, we know that the term $ab + a + b$ above is an integer, because the integers are closed under addition and multiplication (this is another axiom we have.)
(3) the definition of an odd number, again: from (2), we know that $xy = 2(\text{an integer}) + 1$, and is thus odd by definition. So we're done!

□

So, these are pretty much what proofs are like! Notice how we avoided the following two pitfalls in the above proofs:

(1) **Using empirical reasoning.** If you ask the average person to prove something like "The sum of two even numbers are even," they'll often say something like "That's just how they work! Like, $8 + 8 = 16, 4 + 32 = 36$." This is **not a proof**! I cannot reiterate this enough – **examples do not prove things.** They give you intuition, can help you get a feel for what's going on, and can disprove certain things – but they do not prove a statement.
(2) **Not using words.** A common misconception of mathematics is that it just consists of long strings of equations occasionally joined by logical constructions; fifteen-page integrals and the such. Thankfully, that's completely rubbish. Mathematical proofs are arguments, and as such use lots and lots of words! Paragraphs of words, explaining to the reader what's going on and what the author is doing. In the proofs above, lots of verbiage is used to indicate to the reader what is going on – in your proofs, you should do the same! Words are your friends.

6. Proofs by Contradiction

So, the proofs we just did were pretty much straightforward. Sometimes, however, it can be easier to prove something using tricker methods! In this section, we illustrate one such method: the Proof by Contradiction.

What does it mean to prove something by contradiction? Well: consider some proposition P. By definition, P is either true or false. Suppose we want to show that P is true; how could we do this?

One method is to do the following: Take the proposition $\neg P$. If we can show that there is some sentence Q such that $\neg P$ implies Q and that $\neg p$ implies $\neg Q$, then we have that whenever $\neg P$ holds, we have a contradiction! As mathematics is free of contradictions[1], we have that $\neg P$ cannot be true! In other words, we must have that $\neg P$ is false – i.e. that P is true!

If that's too abstract/confusing, consider the following example:

Lemma 6.1. *There is a pair of irrational numbers a, b such that a^b is rational.*

[1]well, it's hopefully free of contradictions: if you're curious, look up Gödel's incompleteness theorem on Wikipedia (or come talk to me!) for some reasons about why this is a little complicated.

Proof. Let's try to prove this by contradiction!

Examine the negation of our claim, which is the following: "For any pair of irrational numbers a, b, a^b is always irrational." If we can show that proposition leads to a contradiction, we know that it can't be true! – and thus that our original claim (that a pair of irrational numbers a, b exist such that a^b is rational) is true.

So: suppose that for any pair of irrational numbers a, b, a^b is always irrational. Then, we know that (in specific)

$$\sqrt{2}^{\sqrt{2}}$$

is irrational.

So: we've just shown that $\sqrt{2}^{\sqrt{2}}$ is irrational, and we know $\sqrt{2}$ to be irrational. Then, if we apply our hypothesis again, we get that

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2$$

is irrational. But 2 is clearly rational! A contradiction – so we know that our supposed claim ("for any pair of irrational numbers a, b, a^b is always irrational") must be false, and thus that our original claim ("there is a pair of irrational numbers a and b such that a^b is rational") is true. □

It's worth noting that our proof above didn't give us a pair of irrational numbers! – just that some pair exists. This is the tricky thing about proofs by contradiction: because they're nonconstructive, they can often tell you that a solution exists without actually telling you what it is!