MATH 8, SECTION 1 - MIDTERM REVIEW NOTES

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Abstract. These are the notes from Friday, Oct. 25th's midterm review.

The following is kind of a condensed, "Cliffs-Notes"-style review of the course thus far; here, we list all of the major theorems and results we've covered thus far, and talk a little bit about when we would use these theorems. Basically, this is the last four and a half weeks in one handout; if you want to see *examples* of how these things are actually used, consult the online notes for those particular weeks!

1. Proof Methods

(Relevant lectures: Monday, wk. 1, Wednesday, wk. 1, Friday, wk. 1.)

Basically, you are all – as a class –quite capable with proof methods; so there's not a lot to say here. However, it is worth it to mention the structure of an inductive proof again, as it's been a while since we've used induction (as opposed to direct proofs or proofs by contradiction!), and a lot of people get tripped up on the structure of these things:

1.1. **Proofs by Induction.** Suppose that you have a claim $P(n)$ – a sentence like "$2^n \geq n$", for example. How do we prove that this kind of thing holds by induction? Well: we generally follow the outline below:

Lemma 1.1. *$P(n)$ holds, for all $n \geq k$.*

Proof.

Base case: we prove (by hand) that $P(k)$ holds, for a few base cases.

Inductive step: Assuming that $P(m)$ holds for all $k \leq m < n$, prove that $P(n)$ holds.

Conclusion: $P(n)$ holds for all $n \geq k$. □

2. Sequences

(Relevant lectures: Friday, wk. 2, Monday, wk. 3,)

2.1. **Definitions.** A **sequence** is just an infinite collection of objects $\{a_n\}_{n=1}^{\infty}$ indexed by the natural numbers. The main property that we've studied about sequences in this class is that of convergence:

Definition 2.1. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value λ if, for any distance ϵ, the a_n's are eventually within ϵ of λ. To put it more formally, $\lim_{n\to\infty} a_n = \lambda$ iff for any distance ϵ, there is some cutoff point N such that for any n greater than this cutoff point, a_n must be within ϵ of our limit λ.

In symbols:

$$\lim_{n\to\infty} a_n = \lambda \text{ iff } (\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

2.2. **Tools.** We have the following tools for manipulating and studying sequences:

(1) **Arithmetic and Sequences**:
- *Additivity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$.
- *Multiplicativity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
- *Quotients of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$.

When using these properties, **please remember** to show that both of the limits $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ exist before splitting them apart! TAs will dock you mad points for failing to do this, as it is one of the most common ways for people to make errors in limit calculations.

(2) **Monotone and Bounded Sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded above and nonincreasing, it also converges. This is a useful trick if you've ran into a sequence that you have no idea where it converges to, but just need to show that it goes *somewhere*.

(3) **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l. Basically, whenever you're studying a sequence whose terms are really complex and messed-up, use the squeeze theorem to bound it above and below by something simple you understand – i.e. if you were looking at $a_n = n^{-2}\sin(n)$, you could bound this above and below by $\pm n^{-2}$.

(4) **Cauchy sequences** A sequence is Cauchy[1] iff it converges. For the most part, this doesn't come up in calculations; it's more of a tool for proving other theorems (at least, this is the case for as much calculus as we've covered.)

2.3. **Applications.** Pretty much everything else we've looked at in this class can be thought of as an "application" of sequences: the largest/most obvious example of this, however, is **series**, the subject of the next section:

3. Series

(Relevant lectures: Monday, wk. 3, Wednesday, wk. 3, Friday, wk. 3.)

3.1. **Definitions.** We defined three main concepts in our work with series, which we repeat here:

Definition 3.1. A sequence is called **summable** if the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums

$$s_n := a_1 + \ldots a_n$$

[1]We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m, n \geq N$

$$|a_m - a_n| < \epsilon.$$

You can think of this condition as saying that Cauchy sequences "settle down" in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

converges. If it does, we then call the limit of this sequence the **sum** of the a_n, and denote this quantity by writing

$$\sum_{n=1}^{\infty} a_n.$$

We call such infinite sums **series**.

Definition 3.2. A series $\sum_{n=1}^{\infty} a_n$ converges **absolutely** iff the series $\sum_{n=1}^{\infty} |a_n|$ converges; it converges **conditionally** iff the series $\sum_{n=1}^{\infty} a_n$ converges but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ diverges.

Definition 3.3. For a series of the form[2] $\sum_{n=0}^{\infty} a_n x^n$, we say that the **radius of convergence** of this series is some value $R \in \mathbb{R}$ such that

- if x is a real number such that $|x| < R$, $\sum_{n=0}^{\infty} a_n x^n$ converges, and
- if x is a real number such that $|x| > R$, $\sum_{n=0}^{\infty} a_n x^n$ diverges.

3.2. **Tools.** Direct calculations of series are often rather difficult; consequently, we have developed the following tools for manipulating them:

(1) **Comparison Test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that $0 \leq a_n \leq b_n$, then the following statement is true:

$$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

When to use this test: when you're looking at something fairly complicated that either (1) you can bound above by something simple that converges, like $\sum 1/n^2$, or (2) that you can bound below by something simple that diverges, like $\sum 1/n$.

(2) **Limit Comparison Test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0,$$

then the following statement is true:

$$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Leftrightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

When to use this test: typically, when you see something like a quotient of really complicated polynomials. It works a lot like the normal comparison test, and merits consideration in many of the same situations.

(3) **Alternating Series Test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers such that
  - $\lim_{n\to\infty} a_n = 0$ monotonically, and
  - the a_n's alternate in sign, then

the series $\sum_{n=1}^{\infty} a_n$ converges.

When to use this test: when you have an alternating series.

(4) **Ratio Test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

then we have the following three possibilities:

[2]Such series are called **power series**, because they are a series made out of increasing powers of x^n.

- If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then we have no idea; it could either converge or diverge.

When to use this test: when you have something that is growing kind of like a geometric series: so when you have terms like 2^n or $n!$.

(5) **Root Test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = r,$$

then we have the following three possibilities:

- If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then we have no idea; it could either converge or diverge.

When to use this test: mostly, in similar situations to the ratio test. Basically, if the ratio test fails, there's a small chance that this will work instead.

(6) **Absolute Convergence and Convergence**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} |a_n|$$

converges, then so does

$$\sum_{n=1}^{\infty} a_n.$$

When to use this test: whenever you have a series of terms that are not all of the same sign, and yet aren't strictly alternating. This is pretty much your only tool to deal with mixed-sign series that aren't solved by the alternating series test: so, if you have to deal with anything like $\cos(n)/2^n$, reach for this theorem first.

(7) **Vanishing criterion**: If $\lim_{n\to\infty} a_n \neq 0$, then the series

$$\sum_{n=1}^{\infty} a_n$$

diverges. When to use this theorem: basically, whenever you're looking at a series made out of things that don't converge to 0, like $(-1)^n$.

3.3. **Applications.** The main applications of series are integrals, which we haven't discussed yet; so, for the most part, our study of series has been focused on just calculations and manipulations of specific series (rather than any specific applications.) It is worth recalling the following facts, however:

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{diverges,}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad \text{converges,}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{and thus converges; finally,}$$

$$\sum_{n=1}^{\infty} \frac{1}{x^n} \qquad \text{converges iff } x < 1, \text{ in which case it is equal to } \frac{x}{1-x}.$$

4. Limits and Continuity

(Relevant lectures: Monday, wk. 4, Wednesday, wk. 4, Friday, wk. 4.)

4.1. **Definitions.** In the fourth week of our course, we turned to the study of limits of functions; here, we encountered our first $\epsilon - \delta$ proofs, and began to work with the notion of continuity. We review several key definitions here:

Definition 4.1. If $f : X \to Y$ is a function between two subsets X, Y of $\mathbb{R}$, we say that

$$\lim_{x \to a} f(x) = L$$

if and only if

(1) (vague:) as x approaches a, $f(x)$ approaches L.
(2) (precise; wordy:) for any distance $\epsilon > 0$, there is some neighborhood $\delta > 0$ of a such that whenever $x \in X$ is within δ of a, $f(x)$ is within ϵ of L.
(3) (precise; symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta) \Rightarrow (|f(x) - L| < \epsilon).$$

Definition 4.2. A function $f : X \to Y$ is said to be **continuous** at some point $a \in X$ iff

$$\lim_{x \to a} f(x) = f(a).$$

Definition 4.3. For a function $f : X \to Y$, we say that

$$\lim_{x \to a^+} f(x) = L$$

if and only if

(1) (vague:) as x goes to a from the right-hand-side, $f(x)$ goes to L.
(2) (concrete, symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta \text{ and } x > a) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x \to a^-} f(x) = L$$

if and only if

(1) (vague:) as x goes to a from the left-hand-side, $f(x)$ goes to L.
(2) (concrete, symbols:)

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X, (|x - a| < \delta \text{ and } x < a) \Rightarrow (|f(x) - L| < \epsilon).$$

Definition 4.4. For a function $f : X \to Y$, we say that

$$\lim_{x \to +\infty} f(x) = L$$

if and only if

(1) (vague:) as x goes to "infinity," $f(x)$ goes to L.
(2) (concrete, symbols:)

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x > N) \Rightarrow (|f(x) - L| < \epsilon).$$

Similarly, we say that

$$\lim_{x \to -\infty} f(x) = L$$

if and only if

(1) (vague:) as x goes to "negative infinity," $f(x)$ goes to L.
(2) (concrete, symbols:)

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall x \in X, (x < N) \Rightarrow (|f(x) - L| < \epsilon).$$

4.2. **Tools.** As before, we developed several useful tools and blueprints for dealing with limits, which we review here:

(1) **A blueprint for $\epsilon - \delta$ proofs of limits**: In class, we developed the following "blueprint" that describes a general method for proving that $\lim_{x \to a} f(x) = L$ via an $\epsilon - \delta$ argument. We review this below:
   (a) First, examine the quantity

   $$|f(x) - L|.$$

   Specifically, try to find a simple upper bound for this quantity that depends only on $|x - a|$, and goes to 0 as x goes to a – something like $|x - a| \cdot (\text{constants})$, or $|x - a|^3 \cdot (\text{bounded functions, like } \sin(x))$.
   (b) Using this simple upper bound, for any $\epsilon > 0$, choose a value of δ such that whenever $|x - a| < \delta$, your simple upper bound $|x - a| \cdot (\text{constants})$ is $< \epsilon$. Often, you'll define δ to be $\epsilon/(\text{constants})$, or somesuch thing.
   (c) Plug in the definition of the limit: for any $\epsilon > 0$, we've found a δ such that whenever $|x - a| < \delta$, we have

   $$|f(x) - L| < (\text{simple upper bound depending on } |x - a|) < \epsilon.$$

   Thus, we've proven that $\lim_{x \to a} f(x) = L$, as claimed.
(2) **A blueprint for proving that certain limits do *not* exist**: In class, we proved the following lemma:

   Lemma 4.5. *For any function $f : X \to Y$, we know that $\lim_{x \to a} f(x) \neq L$ iff there is some sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:*
   - *$\lim_{n \to \infty} a_n = L$, and*
   - *$\lim_{n \to \infty} f(a_n) \neq L$, and*

   This lemma makes proving that a function f is discontinuous at some point a remarkably easy:
   - to prove that $\lim_{x \to a} f(x) \neq L$,

- all we have to do is just find *one* sequence $\{a_n\}_{n=1}^{\infty}$ that converges to a, such that $\lim_{n\to\infty} f(a_n) \neq L$ *on that sequence*! Basically, it allows us to work in the world of sequences instead of that of continuity, which can make life a lot easier on us.

(3) **Squeeze theorem**: If f, g, h are functions defined on some interval $I \setminus \{a\}$[3] such that

$$f(x) \leq g(x) \leq h(x), \forall x \in I \setminus \{a\}, \text{ and}$$
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x),$$

then $\lim_{x\to a} g(x)$ exists, and is equal to the other two limits $\lim_{x\to a} f(x), \lim_{x\to a} h(x)$. Basically, use this theorem like the squeeze theorem for sequences: whenever you see something rather complicated and want to take its limit, try to simply bound it above and below by simple things that you can calculate! This is really useful at studying functions that have trigonometric functions as factors (like $x^2 \sin(1/x)$, for example.)

(4) **Limits and arithmetic**: If f, g are a pair of functions such that $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ both exist, then we have the following equalities:

$$\lim_{x\to a} (\alpha f(x) + \beta g(x)) = \alpha \left(\lim_{x\to a} f(x)\right) + \beta \left(\lim_{x\to a} g(x)\right)$$
$$\lim_{x\to a} (f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$
$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \left(\lim_{x\to a} f(x)\right) / \left(\lim_{x\to a} g(x)\right), \text{ if } \lim_{x\to a} g(x) \neq 0.$$

(5) **Limits and composition**: If $f : Y \to Z$ is a function such that $\lim_{y\to a} f(x) = L$, and $g : X \to Y$ is a function such that $\lim_{x\to b} g(x) = a$, then

$$\lim_{x\to b} f(g(x)) = L.$$

Specifically, if both functions are continuous, their composition is continuous.

Basically, between these three theorems and the results we know from class (e^x, sin, cos, and the polynomials are continuous), you should be able to do most of your limit calculations without touching an ϵ or δ.

4.3. **Applications.** There are two key applications of continuity which we have discussed in class: we review them here:

Theorem 4.6. *(Intermediate Value Theorem): If f is a continuous function on $[a, b]$, then f takes on every value between $f(a)$ and $f(b)$ at least once.*

Most uses of this theorem occur when we have a continuous function f that takes on both positive and negative values on some interval; in this case, the intermediate value theorem tells us that this function must have a zero between each pair of sign changes. Basically, when you have a question that's asking you to find zeroes of

[3]The set $X \setminus Y$ is simply the set formed by taking all of the elements in X that are not elements in Y. The symbol $\setminus$, in this context, is called "set-minus", and denotes the idea of "taking away" one set from another.

a function, or to show that a function with prescribed endpoint behavior takes on some other values, the IVT is the way to go.

Theorem 4.7. *(Extremal value theorem:) If $f : X \to Y$ is a continuous function, and X is a closed and bounded subset X of $\mathbb{R}$, then f attains its minima and maxima. In other words, there are values $m, M \in X$ such that for any $x \in X$, $f(m) \leq f(x) \leq f(M)$.*

We didn't really use this theorem too often for direct calculations (as it only tells you that maxima and minima are attained, but not what they are!) – basically, its main use was in proving one of the applications of differentiation, in our next section!

5. Differentiation

(Relevant lectures: Monday, wk. 5, Wednesday, wk. 5.)

For the midterm, you're liable for the material covered on Monday and Wednesday this week (week 5) in class: so you should know what a derivative is and how they relate to finding minima and maxima, but you don't need to worry about the mean value theorem.

5.1. Definitions.

Definition 5.1. For a function f defined on some neighborhood $(a-\delta, a+\delta)$, we say that f is **differentiable** at a iff the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

exists. If it does, denote this limit as $f'(a)$; we will often call this value the **derivative** of f at a.

The derivative has a number of interpretations as physical phenomena; notably, if $f(x)$ is a function that calculates distance with respect to some time t, you can think of the derivative $f'(t)$ as denoting the velocity of f at time t, and $f''(t)$ as denoting the acceleration of f at time t.

5.2. Tools.

(1) **Differentiation is linear**: For f, g a pair of functions differentiable at a and α, β a pair of constants,

$$(\alpha f(x) + \beta g(x))'\Big|_a = \alpha f'(a) + \beta g'(a).$$

(2) **Product rule**: For f, g a pair of functions differentiable at a,

$$(f(x) \cdot g(x))'\Big|_a = f'(a) \cdot g(a) + g'(a) \cdot f(a).$$

(3) **Quotient rule**: For f, g a pair of functions differentiable at a, $g(a) \neq 0$, we have

$$\left(\frac{f(x)}{g(x)}\right)'\Bigg|_a = \frac{f'(a) \cdot g(a) - g'(a) \cdot f(a)}{(g(a))^2}$$

(4) **Chain rule**: For f a function differentiable at $g(a)$ and g a function differentiable at a,

$$(f(g(x)))'\Big|_a = f'(g(a)) \cdot g'(a).$$

5.3. **Applications.** There's only one application of derivatives that we've really discussed in-depth thus far in the course, and that is the concept of **extrema**:

Definition 5.2. For a function f defined on some set X, the **critical points** of f on X are all of the points in x where either

- $f'(x) = 0$, or
- $f'(x)$ doesn't exist.

Definition 5.3. For a function f defined on some interval (a, b), a point $x \in (a, b)$ is called a **local maxima** iff there is some small neighborhood $(x - \delta, x + \delta)$ in which $f(x) > f(t), \forall t \in (x - \delta, x + \delta)$.

Similarly, a point $x \in (a, b)$ is called a **local minima** iff there is some small neighborhood $(x - \delta, x + \delta)$ in which $f(x) < f(t), \forall t \in (x - \delta, x + \delta)$.

Proposition 5.4. *If x is a local minima or maxima for some function f, x is a critical point of f.*

Proposition 5.5. *If f is a continuous function on some interval $[a, b]$, then f takes on its minima and maxima over this entire region by the extremal value theorem. Furthermore, f takes on these minimum and maximum values at either the critical points of f or at the endpoints $\{a, b\}$.*

The above two propositions are fantastically useful when it comes to analyzing a function's critical points: all we have to do is take the derivative, find all of the critical points, and look at the function at all of those points along with the endpoints.