Math 1d	Instructor: Padraic Bartlett
Lectures 5-6: Taylor Series	
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1 Taylor Polynomials and Series

As we saw in week 4, power series are remarkably **nice** objects to work with. In particular, because power series uniformly converge within their radii of convergence R, we know that the equations

$$\int \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \int a_n x^n dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \text{ and}$$
$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(a_n x^n\right) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

hold and converge for any power series $(\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R, and any $x \in (-R, R)$.

In other words, power series are easily differentiated and integrated as many times as we like! This is remarkably useful: as you probably remember from Math 1a, integration and differentiation are often really difficult tasks.

A natural question is the following: how can we use this observation about power series to work with other kinds of functions? In other words: if I give you a function, is there any way to turn it into a power series, so that we can integrate and differentiate it easily?

As it turns out, there is! As a motivating example, consider the function $f(x) = e^x$. How can we approximate this with a power series?

Initially, we have no idea; so let's try something simpler! Specifically, let's try approximating it at exactly one point; say $x = 0$. Additionally, let's try not approximating it by a power series just yet, but instead try approximating it with just polynomials; maybe being finite will help illustrate what we're trying to do.

Let's start with a polynomial of degree 0 – i.e. a **constant**. What constant best approximates e^x at 0? Well: if we set our constant equal to $e^0 = 1$, then the constant function will at least agree with e^x at 0. That's about as good as we can expect to do: so let's call the **0^{th}-order approximation to** e^x, $T_0(e_x)$, the constant function 1.

Now, let's take a polynomial of degree 1: i.e. a **line**, of the form $T_1(e^x) = a + bx$. What should our constants a and b be? Well: again, if we want this approximation to agree with e^x at 0, we should set $a = 1$, so that $e^0 = 1 = 1+b\cdot 0$. For b; visually, we'd get an even better approximation of e^x if we picked b such that the slope of $a+bx$ and e^x both agreed at 0! In other words, if we set $b = 1$, we would have $\frac{d}{dx}\left(e^x\right)\Big|_{x=0} = e^x\Big|_{x=0} = 1 = \frac{d}{dx}\left(1 + 1 \cdot x\right)\Big|_{x=0}$. So, let's define the **1^{st}-order approximation to** e^x, $T_1(e_x)$, the line $1 + x$.

Similarly, to make the **2^{nd}-order approximation to** e^x, $T_2(e_x)$, we want to take a degree-2 polynomial $a_0 + a_1 x + a_2 x^2$ and find values of a_0, a_1, a_2 such that the 0^{th}, 1^{st}, and

2^{nd} derivatives of e^x agree with $T_2(e^x)$ at 0. Therefore, we have

$$\begin{aligned}
(e^x)\Big|_{x=0} &= 1 = \left(a_0 + a_1 x + a_2 x^2\right)\Big|_{x=0} = a_0 \\
&\qquad\qquad \Rightarrow a_0 = 1, \\
\frac{d}{dx}(e^x)\Big|_{x=0} = (e^x)\Big|_{x=0} &= 1 = \frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2\right)\Big|_{x=0} = a_1 \\
&\qquad\qquad \Rightarrow a_1 = 1, \\
\frac{d^2}{dx^2}(e^x)\Big|_{x=0} = (e^x)\Big|_{x=0} &= 1 = \frac{d^2}{dx^2}\left(a_0 + a_1 x + a_2 x^2\right)\Big|_{x=0} = 2a_2 \\
&\qquad\qquad \Rightarrow a_2 = \frac{1}{2},
\end{aligned}$$

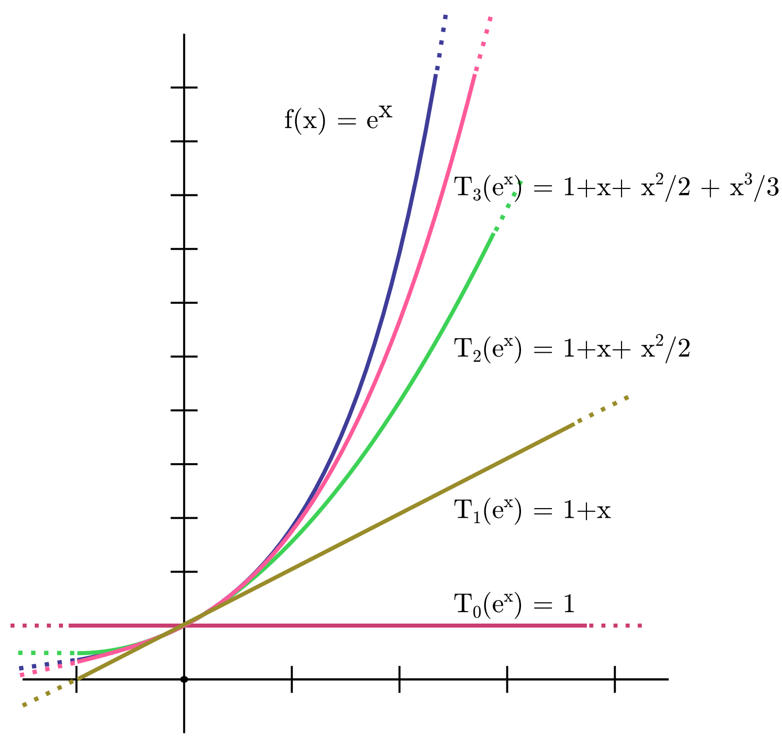
and therefore that the 2^{nd}-order approximation to e^x, $T_2(e^x)$, is $1 + x + \frac{x^2}{2}$.

Using similar logic, we can see that because

$$\begin{aligned}
\frac{d^n}{dx^n}(e^x)\Big|_{x=0} &= (e^x)\Big|_{x=0} = 1, \text{ and} \\
\frac{d^n}{dx^n}\left(a_0 + a_1 x + \ldots a_n x^n\right)\Big|_{x=0} &= \frac{d^{n-1}}{dx^{n-1}}\left(a_1 + 2a_2 x + \ldots n a_n x^{n-1}\right)\Big|_{x=0} \\
&= \frac{d^{n-2}}{dx^{n-2}}\left(2a_2 x + 3 \cdot 2 a_3 x + \ldots (n) \cdot (n-1) a_n x^{n-2}\right)\Big|_{x=0} \\
\ldots &= n! a_n,
\end{aligned}$$

in general we have $a_n = \frac{1}{n!}$, and thus that the n^{th}-degree approximation to e^x is $\sum_{n=0}^{n} \frac{x^n}{n!}$.

We graph some of these approximations here:



Surprisingly, these "local" approximations are getting close to e^x not just at 0, but in fact in a decently-sized neighborhood of 0! In fact, if we take the limit of these approximations

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

this power series turns out to be **equal** to e^x everywhere!

So: this idea of local approximations appears to be giving us a bunch of global data! This motivates the following definition of a Taylor polynomial and Taylor series, which we give here:

Definition 1.1. Let $f(x)$ be a n-times differentiable function on some neighborhood $(a-\delta, a+\delta)$ of some point a. We define the $\mathbf{n^{th}}$ **Taylor polynomial of** $f(x)$ **around** a as the following degree-n polynomial:

$$T_n(f(x), a) := \sum_{n=0}^{n} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n.$$

Notice that this function's first n derivatives all agree with $f(x)$'s derivatives: i.e. for any $k \leq n$,

$$\frac{\partial^k}{\partial x^k} \left(T_n(f(x), a)\right)\Big|_a = f^{(k)}(a).$$

This motivates the idea of these Taylor polynomials as " n^{th} order approximations at a" of the function $f(x)$: if you only look at the first n derivatives of this function at a, they agree with this function completely.

We define the $\mathbf{n^{th}}$ **order remainder function of** $f(x)$ **around** a as the difference between $f(x)$ and its n^{th} order approximation $T_n(f(x), a)$:

$$R_n(f(x), a) = f(x) - T_n(f(x), a).$$

If f is an infinitely-differentiable function, and $\lim_{n\to\infty} R_n(f(x), a) = 0$ at some value of x, then we can say that these Taylor polynomials converge to $f(x)$, and in fact write $f(x)$ as its **Taylor series**:

$$T(f(x)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n.$$

Usually, we will assume that our Taylor series are being expanded around $a = 0$, and omit the a-part of the expressions above. If it is not specified, always assume that you're looking at a Taylor series expanded around 0.

One of the largest questions, given a function $f(x)$, is the following: at which values of x is $f(x)$ equal to its Taylor series? Equivalently, our question is the following: for what values of x is $\lim_{n\to\infty} R_n(f(x)) = 0$?

Our only/most useful tool for answering this question is the following theorem of Taylor:

Theorem 1.2. *(Taylor's theorem:) If $f(x)$ is infinitely differentiable, then*

$$R_n(f(x), a) = \int_a^x \frac{\frac{d^{n+1}}{dx^{n+1}}\left(f(x)\right)\Big|_{x=t}}{n!} \cdot (x-t)^n dt.$$

In other words, we can express the remainder (a quantity we will often not understand) as an integral involving the derivatives of f divided by $n!$ (which is often easily bounded) and polynomials (which are also easy to deal with.)

The main use of Taylor series, as stated before, is our earlier observation about how easy it is to integrate and differentiate power series:

Theorem 1.3. *Suppose that $f(x)$ is a function with Taylor series*

$$T(f(x)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n,$$

and furthermore suppose that $f(x) = T(f(x))$ on some interval $(-a, a)$. Then we can integrate and differentiate $f(x)$ by just termwise integrating and differentiating $T(f(x))$: i.e.

$$\frac{d}{dx} f(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{f^{(n)}(a)}{n!} \cdot (x-a)^n\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{(n-1)!} \cdot (x-a)^{n-1}, \text{ and}$$
$$\int f(x)dx = \sum_{n=0}^{\infty} \int \left(\frac{f^{(n)}(a)}{n!} \cdot (x-a)^n\right) dx = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{(n+1)!} \cdot (x-a)^{n+1} + C.$$

In the following section, we study several functions, find their Taylor series, and find out where these Taylor series converge to their original functions:

2 Taylor Series: Examples

Proposition 2.1. *The Taylor series for $f(x) = e^x$ about 0 is*

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Furthermore, this series converges and is equal to e^x on all of $\mathbb{R}$.

Proof. From our discussion earlier, we know that

$$T(e^x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Furthermore, by using Taylor's theorem, we know that the remainder term $R_n(e^x)$ is just

$$R_n(e^x) = \int_0^x \frac{\frac{d^{n+1}}{dx^{n+1}}(e^x)\Big|_{x=t}}{n!} \cdot (x-t)^n dt = \int_0^x \frac{e^t}{n!} \cdot (x-t)^n dt.$$

Integrating this directly seems... painful. However, we don't need to know exactly what this integral is: we just need to know that it gets really small as n goes to infinity! So, instead of calculating this integral directly, we can just come up with some upper bounds on its magnitude.

Specifically: on the interval $[0, x]$, the function $|e^t|$ is bounded above by $e^{|x|}$, and the function $|(x-t)^n|$ takes on its maximum at $t = 0$, where it's $|x|^n$. Therefore, we have that

$$\left| \int_0^x \frac{e^t}{n!} \cdot (x-t)^n dt \right| \leq \int_0^{|x|} \frac{e^{|x|}}{n!} \cdot |x|^n dt.$$

The function being integrated at the right is just a constant with respect to t: there aren't any t's in it! This makes integration a lot easier, as the integral of a constant is just that constant times the length of the interval:

$$\int_0^{|x|} \frac{e^{|x|}}{n!} \cdot |x|^n dt = \left(\frac{e^{|x|}}{n!} \cdot |x|^n \right) \cdot t \Big|_0^{|x|} = e^{|x|} \cdot \frac{|x|^{n+1}}{n!}.$$

Again: to show that e^x is equal to its Taylor series on all of $\mathbb{R}$, we just need to show that the remainder terms $R_n(e^x)$ always go to 0 as n goes to infinity. So, to finish our proof, it suffices to show that

$$\lim_{n\to\infty} e^{|x|} \cdot \frac{|x|^{n+1}}{n!} = 0.$$

This is not hard to see: simply notice that because the ratio of successive terms is just

$$\frac{e^{|x|} \cdot \frac{|x|^{n+1}}{n!}}{e^{|x|} \cdot \frac{|x|^n}{(n-1)!}} = \frac{|x|}{n},$$

as n goes to infinity the ratio between successive terms goes to 0. In specific, for $n > 2|x|$, the ratio between any two consecutive terms is $< \frac{1}{2}$; i.e. each consecutive term in our sequence is at most half as big as the one that preceeded it. Therefore, the sequence converges to 0, as any sequence consisting of numbers that you're repeatedly chopping in half must converge to 0. $\square$

We work a second example here:

Proposition 2.2. *The Taylor series for* $f(x) = \sin(x)$ *about 0 is*

$$\sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Furthermore, this series converges and is equal to $\sin(x)$ *on all of* $\mathbb{R}$.

Proof. By induction, we know that for any n,

$$\begin{aligned}
\frac{\partial^{4n}}{\partial x^{4n}}(\sin(x))\Big|_0 &= \sin(x)\Big|_0 = 0 \\
\frac{\partial^{4n+1}}{\partial x^{4n+1}}(\sin(x))\Big|_0 &= \cos(x)\Big|_0 = 1 \\
\frac{\partial^{4n+2}}{\partial x^{4n+2}}(\sin(x))\Big|_0 &= -\sin(x)\Big|_0 = 0 \\
\frac{\partial^{4n+3}}{\partial x^{4n+3}}(\sin(x))\Big|_0 &= -\cos(x)\Big|_0 = -1.
\end{aligned}$$

Consequently, we have that the $2n+1$-st Taylor polynomial for $\sin(x)$ around 0 is just

$$T_{2n+1}(\sin(x),0) = \sum_{k=0}^{n} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

because plugging in the derivatives of $\sin(x)$ into the formula for Taylor polynomials just kills off every other term in the sum, and therefore that the Taylor series for $\sin(x)$ is

$$\sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Where does this series converge to $\sin(x)$? Well: if we look at the $R_n(\sin(x))$ terms, we have by Taylor's theorem that

$$R_n(\sin(x)) = \int_0^x \frac{\frac{d^{n+1}}{dx^{n+1}}\left(\sin(x)\right)_{x=t}}{n!} \cdot (x-t)^n dt.$$

Like before, we can simplify this expression by replacing several terms with upper bounds. For example, we can replace $\frac{d^{n+1}}{dx^{n+1}}\left(\sin(x)\right)_{x=t}$ with 1, because the derivatives of $\sin(x)$ are all either $\pm\sin(x)$ or $\pm\cos(x)$, and in either case are bounded above in magnitude by 1; as well, we can bound $|(x-t)|^n$ above by $|x|^n$, for $t \in [0,x]$. This gives us

$$|R_n(\sin(x))| \leq \int_0^{|x|} \frac{1}{n!} \cdot |x|^n dt = \frac{|x|^{n+1}}{n!},$$

which goes to 0 as n goes to infinity, for any value of x. So $\sin(x)$ is equal to its Taylor series on all of $\mathbb{R}$, as claimed! $\square$

Given our success rate so far with approximating functions by Taylor series, you might wonder why we bother with the $R_n(f(x))$-part of our proofs. The following example illustrates why it is that we *do* have to check that a Taylor series converges to its original function; as it turns out, it's possible for a function to have a Taylor series that is completely useless for approximating it away from 0!

Example 2.3. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then $T(f(x)) = 0$, and $T(f(x)) = f(x)$ only at $x = 0$.

Proof. So: derivatives of e^{-1/x^2} are tricky to directly calculate. However, we don't have to! Simply notice that we have

$$\begin{aligned} \frac{d}{dx}\left(e^{-1/x^2}\right) &= \frac{2}{x^3} \cdot e^{-1/x^2}, \text{ and} \\ \frac{d}{dx}\left(\left(\frac{\text{polynomials}}{\text{polynomials}}\right) \cdot e^{-1/x^2}\right) &= \left(\frac{\text{polynomials}}{\text{polynomials}}\right)' \cdot e^{-1/x^2} + \left(\frac{\text{polynomials}}{\text{polynomials}}\right) \cdot \frac{2}{x^3} \cdot e^{-1/x^2} \\ &= \left(\frac{\text{polynomials}}{\text{polynomials}}\right) \cdot e^{-1/x^2}. \end{aligned}$$

In other words, all of the derivatives of e^{-1/x^2} look like some ratio of polynomial expressions, multiplied by e^{-1/x^2}. So, when we look at the limit as x goes to 0 of derivatives of e^{-1/x^2}, we always have

$$\lim_{n\to\infty}\left(\frac{\text{polynomials}}{\text{polynomials}}\right)\cdot e^{-1/x^2}=0,$$

because the exponential term dominates (i.e. exponentials shrink faster than polynomials can grow.)

So, while we don't explicitly know the derivatives of $f(x)$, we **do** know that they're all defined and equal to 0 at $x=0$! So, the Taylor series of $f(x)$ is just

$$\sum_{k=0}^{\infty}\frac{0}{n!}\cdot x^n=0,$$

which is only equal to $f(x)$ at 0, because $e^{-1/x^2}\neq 0$ for any $x\neq 0$. $\square$

Using similar techniques to the worked examples above, you can prove that the functions below have the following Taylor series, and furthermore that they converge to their Taylor series on the claimed sets:

Proposition 2.4.

$$T(\cos(x))=\sum_{k=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},\ \textit{and}\ T(\cos(x))=\cos(x)\ \textit{whenever}\ x\in\mathbb{R}.$$

$$T\left(\frac{1}{1-x}\right)=\sum_{k=0}^{\infty}x^n,\ \textit{and}\ T\left(\frac{1}{1-x}\right)=\frac{1}{1-x}\ \textit{whenever}\ x\in(-1,1).$$

In addition, by substituting terms like $-x^2$ into the above Taylor series, we can derive Taylor series for other functions:

Proposition 2.5.

$$T(e^{-x^2})=\sum_{k=0}^{\infty}(-1)^n\frac{x^{2n}}{n!},\ \textit{and}\ T(e^{-x^2})=e^{-x^2}\ \textit{whenever}\ x\in\mathbb{R}.$$

$$T\left(\frac{1}{1+x^2}\right)=\sum_{k=0}^{\infty}(-1)^n x^{2n},\ \textit{and}\ T\left(\frac{1}{1+x^2}\right)=\frac{1}{1+x^2}\ \textit{whenever}\ x\in(-1,1).$$

2.1 Applications of Taylor Series

Using Taylor series, we can approximate integrals that would otherwise be impossible to deal with. For example, consider the **Gaussian integral**:

$$\int e^{-x^2}dx.$$

Somwhat frustratingly, there is no "elementary" antiderivative for e^{-x^2}: in other words, there is no finite combination of functions like $\sin(x), e^x, x^n$ that will give an antiderivative of e^{-x^2}. Using Taylor series – i.e. an infinite number of functions – we **can** find this integral, to any level of precision we desire! We outline the method below:

Question 2.6. *Approximate*

$$\int_{-1/2}^{1/2} e^{-x^2} dx$$

to within $\pm .0001$ *of its actual value.*

Proof. Above, we proved that

$$T_n(e^x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Using this, we can write

$$e^{-x^2} = T_n(e^{-x})\Big|_{x^2} + R_n(e^{-x})\Big|_{x^2},$$

and therefore write

$$\int_0^2 e^{-x^2} dx = \int_0^2 T_n(e^{-x})\Big|_{x^2} dx + \int_0^2 R_n(e^{-x})\Big|_{x^2} dx.$$

Why is this nice? Well: the T_n part is just a polynomial: specifically, we have

$$T_n(e^{-x})\Big|_{x^2} = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{k!},$$

which is quite easy to integrate! As well, the R_n-thing is something that should be rather small for large values of n: so in theory we should be able to make its integral small, as well!

Specifically: using Taylor's theorem and the estimates we came up with earlier in our lectures, we have

$$\begin{aligned}
|R_n(e^x)| &= \left| \int_0^x \frac{e^t}{n!} \cdot (x-t)^n dt \right| \\
&\leq \int_0^{|x|} \frac{e^{|x|}}{n!} \cdot |x-t|^n dt \\
&\leq \int_0^{|x|} \frac{e^{|x|}}{n!} \cdot |x|^n dt \\
&= \frac{e^{|x|} \cdot |x|^{n+1}}{n!} \\
\Rightarrow \left| R_n(e^x)\Big|_{-x^2} \right| &\leq \frac{e^{x^2} \cdot |x|^{2n+2}}{n!}.
\end{aligned}$$

Using this, we can bound the integral of our remainder terms:

$$\begin{aligned}
\left|\int_{-1/2}^{1/2} R_n(e^{-x})\Big|_{x^2} dx\right| &\leq 2\cdot\int_0^{1/2}\frac{e^{x^2}\cdot x^{2n+2}}{n!}dx\\
&\leq 2\cdot\int_0^{1/2} e^{1/4}\cdot\frac{x^{2n+2}}{n!}dx\\
&= 2e^{1/4}\cdot\left(\frac{x^{2n+3}}{(2n+3)\cdot n!}\right)\Big|_0^{1/2}\\
&= \frac{e^{1/4}}{2^{2n+2}\cdot(2n+3)\cdot n!}
\end{aligned}$$

This quantity is $\leq .0001$ at $n=3$. Therefore, we know that

$$\int_{-1/2}^{1/2} e^{-x^2}dx = \int_{-1/2}^{1/2} T_3(e^x)\Big|_{x^2} dx,$$

up to $\pm.0001$.

So: to find this integral, it suffices to integrate $T_3(e^x)\Big|_{x^2}$. This is pretty easy:

$$\begin{aligned}
\int_{-1/2}^{1/2} T_3(e^x)\Big|_{x^2} dx &= \int_{-1/2}^{1/2}\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{6}\right)dx\\
&= 2\cdot\int_0^{1/2}\left(1-x^2+\frac{x^4}{2}-\frac{x^6}{6}\right)dx\\
&= 2\cdot\left(x-\frac{x^3}{3}+\frac{x^5}{10}-\frac{x^7}{42}\right)\Big|_0^{1/2}\\
&= \frac{4133}{4480}\\
&\cong .922545\ldots
\end{aligned}$$

So this is the integral of e^{-x^2}, up to $\pm.0001$. Using Mathematica, we can see that the actual integral is $\cong .922562$, which verifies that our calculations worked! $\square$

We close with one last application of Taylor series to the calculation of limits, as an alternate strategy to repeated applications of L'Hôpital's rule:

Example 2.7. Find the limit

$$\lim_{n\to\infty}\frac{\sin(x^2)-x^2+\frac{x^6}{6}}{x^{10}}$$

Proof. One way we could solve this problem would be to hit it with L'Hôpital's rule ten times in a row, and get that we're looking at the limit as x goes to 0 of the fraction

$$\frac{30240\cos(x^2)-403200x^4\cos(x^2)+23040x^8\cos(x^2)-302400x^2\sin(x^2)+161280x^6\sin(x^2)-1024x^{10}\sin(x^2)}{10!},$$

which is just $\frac{30420}{10!} = \frac{1}{120}$. And, you know, if you wanted to calculate ten derivatives in a row, you could do that.

I ... don't. So: what's a non-awful way to do this? Well: if we used $\sin(x)$'s Taylor series, we can notice that

$$\sin(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{6} + \frac{x^{10}}{5!} - \dots,$$

and therefore that the fraction we're working with is just

$$\begin{aligned}\frac{\sin(x^2) - x^2 + \frac{x^6}{6}}{x^{10}} &= \frac{\left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{5!} - \dots\right) - x^2 + \frac{x^6}{6}}{x^{10}} \\ &= \frac{1}{x^{10}} \cdot \left(\frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \dots\right) \\ &= \frac{1}{5!} + \left(\frac{x^4}{7!} - \frac{x^8}{9!} + \frac{x^{12}}{11!} - \dots\right)\end{aligned}$$

So: what does this do when x goes to 0? Well, the expression on the right is a power series that converges on all of $\mathbb{R}$, if we use absolute convergence $\Rightarrow$ convergence along with the ratio test. Therefore, it's a **continuous** function! So, to find the limit, we just need to plug in 0:

$$\begin{aligned}\lim_{x \to 0} \frac{\sin(x^2) - x^2 + \frac{x^6}{6}}{x^{10}} &= \left(\frac{1}{5!} + \left(\frac{x^4}{7!} - \frac{x^8}{9!} + \frac{x^{12}}{11!} - \dots\right)\right)\Bigg|_{x=0} \\ &= \frac{1}{5!} \\ &= \frac{1}{120}.\end{aligned}$$

Which worked! And we didn't have to calculate ten derivatives. Nice.

$\square$