Math 1d Instructor: Padraic Bartlett

# Homework 2

*Due date: Wednesday, February 6.* *Caltech 2013*

**Instructions**: There are **two** sections to this homework. In the first section, complete **all three problems** listed. In the second section, choose **two** out of the listed seven to complete. If you find yourself stuck, frustrated, or spending much more than six hours on the total problem set, contact me!

For the first section: the only resources allowed are your own notes, the online notes, and textbooks. Collaboration is not allowed on these three problems.

For the second section: you are allowed to use Wikipedia, Mathematica, your notes, the online class notes, textbooks, your classmates, and other Caltech students. All of your work should, however, be written up in your own words, and you should understand your proofs well (i.e. if someone else were to ask you how to do this problem, you should be able to teach them your solution and persuade them that your methods are correct.) If you've completed a problem via Mathematica or writing a program, **you must attach your code** to receive credit.

**Section One**:

1. Determine whether the following series converge. Prove your claims.

$$\text{(a)} \qquad \sum_{n=1}^{\infty} \frac{\sin(n)}{e^n}.$$

$$\text{(b)} \qquad \sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

$$\text{(c)} \qquad \sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(n)}.$$

2. Define the function $f_n$ as follows:

$$f_n(x) = \begin{cases} 0, & x \leq 0, \\ x^n - x^{2n}, & 0 < x < 1, \\ 0, & 1 \leq x, \end{cases}$$

   (a) Show that the pointwise limit $\lim_{n\to\infty} f_n$ of these functions is the zero function $f(x) = 0$.

   (b) Show that these functions do not converge uniformly to 0. (Hint: using the derivative, find the maxima of $f_n$. What is this function equal to at its maxima? Why is this a problem if the functions are supposed to be converging uniformly to 0?)

3. Find all of the values of $x \in \mathbb{R}$ on which the following power series converge. Prove your claims.

$$\text{(a)} \qquad \sum_{n=1}^{\infty} \frac{x^n}{n \ln(n)}.$$

$$\text{(b)} \qquad \sum_{n=1}^{\infty} n^n \cdot x^n.$$

$$\text{(c)} \qquad \sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(n)} x^n.$$

**Section Two**:

1. (Dynamical Systems!): Suppose that $f(x)$ is a continuous function on $\mathbb{R}$ with the following property:

$$\forall a, b \in \mathbb{R}, |f(a) - f(b)| < \frac{1}{2}|a - b|.$$

Examples of such functions $f$: $f(x) = x/2, f(x) = \sin(x)/2, \ldots$ You can kind of think of $f(x)$ as a function that "compresses space."

Let $\{f_n(x)\}_{n=1}^{\infty}$ be defined as follows:

$$f_1(x) = f(x),$$
$$f_{n+1}(x) = f(f_n(x)).$$

In other words, $f_n(x)$ is just the function created by composing $f$ with itself $n$ times.

(a) Using induction, show that for any value $x$, we have

$$|f_{n+1}(x) - f_n(x)| \leq \frac{1}{2^n} \cdot |f(x) - x|.$$

(b) Using a telescoping sum, write

$$f_{n+1}(x) - f_1(x) = \sum_{k=1}^{n} f_{k+1}(x) - f_k(x).$$

(c) Use this observation, part (a), and your knowledge of series to show that

$$\lim_{n \to \infty} f_{n+1}(x)$$

exists and is finite. Denote the value of this limit as $x_0$,

(d) Show that $f(x_0) = x_0$: i.e. that $x_0$ is a **fixed point** of $f$. (Hint: apply $f$ to both sides of your limit, and use the fact that $f$ is continuous.)

2. In class, we rearranged the terms of the series$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ into the series

$$\begin{aligned}
&\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}\cdots\\
=&\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}\cdots\\
=&\frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.
\end{aligned}$$

(a) Using similar techniques, rearrange and group/split up the terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ to get a series that converges to

$$2\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

(b) Now, create a rearrangement of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ into a series that **diverges**. (Hint: use the fact that the harmonic series diverges to show that if you add up a bunch of terms all with the same sign, you can get arbitrarily large sums. Use this to describe a rearrangement into a series that cannot converge.)

3. In class, we proved that the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. Using the ratio test and the identity $\lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e}$, show that the series

$$\sum_{n=1}^{\infty}\frac{a^n\cdot n!}{n^{n+1}}$$

converges for $a < e$, and diverges for $a > e$.

4. Recall, from the first problem set, the Fibonacci sequence:

$$f_0 = 0, f_1 = 1, f_{n+1} = f_n + f_{n-1}.$$

Look at the following power series you get by using the Fibonacci sequence, starting at $f_1$, as the coefficients of the $x^n$'s: i.e.

$$\begin{aligned}
P(x) =&\sum_{n=0}^{\infty} f_{n+1}\cdot x^n\\
=&1+1x+2x^2+3x^3+5x^4+8x^5+13x^6+21x^7\ldots
\end{aligned}$$

(a) Show that this power series is equal to the function

$$f(x)=\frac{1}{1-x-x^2}.$$

(Hint: Notice that $f(x)$ is defined by the equation $f(x)-xf(x)-x^2f(x)=1$.)

(b) Factor $1 - x - x^2$ into its roots. What do you get? Is it related to the formula

$$f_n = \frac{\varphi^n - (-1/\varphi)^n}{\sqrt{5}},$$

that we derived on our first problem set? (An informal discussion is OK here; not looking for a proof, but rather if you have any ideas on how these things are related. Grading will be entirely on (a) and finding the roots for (b.))

5. In class, we said that you cannot find a sequence of continuous functions that converge uniformly to a discontinuous function. Can you find a sequence of discontinuous functions that converge uniformly to a continuous function? Either construct an example that shows that you can, or prove that you cannot.

6. Suppose that $\sum_{n=1}^{\infty} a_n x^n$ is a power series such that $\sum_{n=1}^{\infty} a_n x^n = 0$, for any $x \in \mathbb{R}$. Prove that $a_n = 0$, for every $n$.

7. Suppose you have a $\mathbb{Z} \times \mathbb{Z}$ grid of squares. Consider the following game we can play on this board:

   - Starting position: place one coin on every single square below the $x$-axis.
   - Moves: If there are two coins in a row (horizontally or vertically) with an empty space ahead of them, you can "jump" either one of the coins over the other – i.e. you can remove those two coins and put a new coin on the space directly ahead of them.

   Either get a coin above the green line through a sequence of moves (attach and describe these moves,) or prove that you cannot get a coin above the green line in finitely many moves. (Hint: use a technique similar to the "weighting" trick we described for the coin game on the first problem set.)