Math 1d	Instructor: Padraic Bartlett

Lecture 6: Complex Numbers and Series

Week 7 — *Caltech - Winter 2012*

1 Random Questions

Question 1.1. *Show that in any group of 6 people, there are either three mutual acquaintances*[1] *or three mutual strangers.*

(Open:) Find the smallest value of n such that in any group of n people, there are either 5 mutual acquaintances or 5 mutual strangers.

Question 1.2. *In class, we formed the complex numbers $\mathbb{C}$ by taking $\mathbb{R}$ and adding the symbol i, where we thought of i as $\sqrt{-1}$.*

Similarly, we can define the ***Gaussian integers*** *as the set*

$$\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\},$$

formed by taking the integers and adding the symbol i.

Show that there is no set of three points $\alpha, \beta, \gamma \in \mathbb{Z}[i]$ such that these three points form the vertices of an equilateral triangle.

2 The Structure of the Complex Numbers

In Ma1a, we often ran into the following question: "Given some polynomial $P(x)$, what are its roots?" Depending on the polynomial, we had several techniques for finding these roots (Rolle's theorem, quadratic/cubic formulas, factorization;) however, we at times would encounter polynomials that have no roots at all, like

$$x^2 + 1.$$

Yet, despite the observation that this polynomial's graph never crossed the x-axis, we *could* use the quadratic formula to find that this polynomial had the "formal" roots

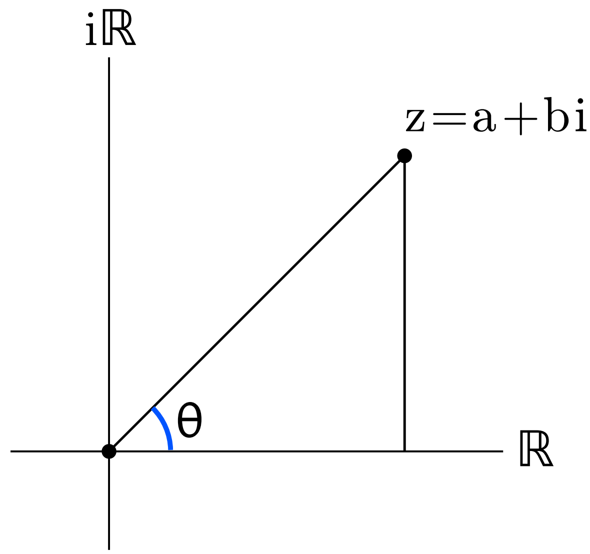
$$\frac{-0 \pm \sqrt{-4}}{2} = \pm\sqrt{-1}.$$

The number $\sqrt{-1}$, unfortunately, isn't a real number (because $x^2 \geq 0$ for any real x, as we proved last quarter) – so we had that this polynomial has no roots over $\mathbb{R}$. This was a rather frustrating block to run into; often, we like to factor polynomials entirely into their roots, and it would be quite nice if we could always do so, as opposed to having to worry about irreducible functions like $x^2 + 1$.

Motivated by this, we can create the **complex numbers** by just throwing $\sqrt{-1}$ into the real numbers. Formally, we define the set of complex numbers, $\mathbb{C}$, as the set of all numbers $\{a + bi : a, b \in \mathbb{R}\}$, where $i = \sqrt{-1}$.

[1]For the purposes of this question, assume that for any pair of people, either they have both met each other (they're both acquainted with each other) or neither one has met the other (they are mutual strangers.) I.e. there is no situation where one person has met the other but the other somehow didn't notice.

Graphically, we can visualize the complex numbers as a plane, where we identify one axis with the real line $\mathbb{R}$, the other axis with the imaginary-real line $i\mathbb{R}$, and map the point $a+bi$ to (a,b):



Two useful concepts when working in the complex plane are the ideas of **norm** and **conjugate**:

Definition 2.1. If $z = x + iy$ is a complex number, then we define $|z|$, the **norm** of z, to be the distance from z to the origin in our graphical representation; i.e. $|z| = \sqrt{x^2+y^2}$.

As well, we define the **conjugate** of $z = x+iy$ to be the complex number $\overline{z} = x - iy$. Notice that $|z| = \sqrt{x^2+y^2} = \sqrt{z\overline{z}}$.

In the real line, recall that we had $|x \cdot y| = |x| \cdot |y|$; this still holds in the complex plane! In particular, we have $|w \cdot z| = |w| \cdot |z|$, for any pair of complex numbers w, z. (If you don't believe this, prove it! – it's not a difficult exercise to check.)

So: we have this set, $\mathbb{C}$, that looks like the real numbers with i thrown in. Do we have any way of extending any of the functions we know and like on $\mathbb{R}$, like $\sin(x), \cos(x), e^x$ to the complex plane?

At first glance, it doesn't seem likely: i.e. what should we say $\sin(i)$ is? Is cos a periodic function when we add multiples of $2\pi i$ to its input? Initially, these questions seem unanswerable; so (as mathematicians often do when faced with difficult questions) let's try something easier instead!

In other words, let's look at **polynomials**. These functions are much easier to extend to $\mathbb{C}$: i.e. if we have a polynomial on the real line

$$f(x) = 2x^3 - 3x + 1,$$

the natural way to extend this to the complex line is just to replace the x's with z's: i.e.

$$f(z) = 2z^3 - 3z + 1.$$

This gives you a well-defined function on the complex numbers (i.e. you put a complex number in and you get a complex number out,) such that if you restrict your inputs to the real line $x + i \cdot 0$ in the complex numbers, you get the same outputs as the real-valued polynomial.

In other words, we know how to work with polynomials. Does this help us work with more general functions? As we've seen over the last two weeks, the answer here is yes! More specifically, the answer here is to use **power series**. Specifically, over the last week, we showed that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots,$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots, \text{ and}$$
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

for all real values of x. Therefore, we can choose to **define**

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots,$$
$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots, \text{ and}$$
$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots,$$

for all $z \in \mathbb{C}$. This extension has the same properties as the one we chose for polynomials: it gives a nice, consistent definition of each of these functions over all of $\mathbb{C}$, that agrees with the definitions they already had on the real line $\mathbb{R}$.

The only issue with these extensions is that we're still not entirely quite sure what they mean. I.e.: what **is** $\sin(i)$, apart from some strange infinite power series? Where does the point e^z lie on the complex plane?

To answer these questions, let's look at e^z first, as it's arguably the easiest of the three (its terms don't do the strange alternating-thing, and behave well under most algebraic manipulations.) In particular, write $z = x + iy$: then we have

$$e^z = e^{x+iy} = e^x \cdot e^{iy},$$

where e^x is just the real-valued function we already understand. So, it suffices to understand e^{iy}, which we study here:

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \frac{(iy)^7}{7!} + \frac{(iy)^8}{8!} + \ldots$$

If we use the fact that $i^2 = -1$, we can see that powers of i follow the form

$$i, -1, -i, 1, i, -1, -i, 1, \ldots$$

and therefore that

$$e^{iy} = 1 + iy - \frac{y^2}{2} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - \frac{y^6}{6!} - i\frac{y^7}{7!} + \frac{y^8}{8!} + \ldots$$

If we split this into its real and imaginary parts, we can see that

$$e^{iy} = \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} \dots\right).$$

But wait! We've seen those two series before: they're just the series for $\sin(y)$ and $\cos(y)$! In other words, we've just shown that

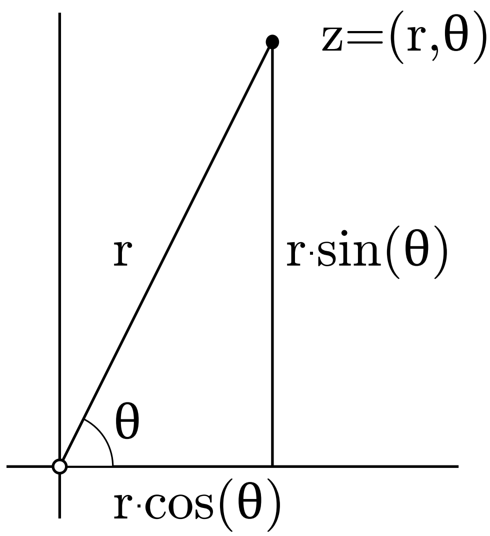
$$e^{iy} = \cos(y) + i\sin(y).$$

One famous special case of this formula is when $y = \pi$, in which case we have $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$, or

$$e^{i\pi} + 1 = 0.$$

Which is *amazing*. In one short equation, we've discovered a fundamental relation that connects five of the most fundamental mathematical constants, in a way that – without this language of power series and complex numbers – would be unfathomable to understand. Without power series, the fact that a constant related to the area of a circle (π), the square root of negative 1, the concept of exponential growth (e) and the multiplicative identity (1) can be combined to get the additive identity (0) would just seem absurd; yet, with them, we can see that this relation was inevitable from the very definitions we started from.

But that's not all! This formula (Euler's formula) isn't just useful for discovering deep fundamental relations between mathematical constants: it also gives you a way to visualize the complex plane! In specific, recall the concept of **polar coördinates**, which assigned to each nonzero point z in the plane a value $r \in \mathbb{R}^+$, denoting the distance from this point to the origin, and an angle $\theta \in [0, 2\pi)$, denoting the angle made between the positive x-axis and the line connecting z to the origin:



With this definition made, notice that any point with polar coördinates (r, θ) can be written in the plane as $(r\cos(\theta), r\sin(\theta))$. This tells us that any point with polar coördinates (r, θ) in the complex plane, specifically, can be written as $r(\cos(\theta) + i\sin(\theta))$; i.e. as $re^{i\theta}$.

This gives us what we were originally looking for: a way to visually interpret e^{x+iy}! In specific, we've shown that e^{x+iy} is just the point in the complex plane with polar coördinates (e^x, y).

2.1 A Quick Interlude: Factorization and $\sum \frac{1}{n^2}$

As alluded to in our "motivation" for the complex numbers, working in $\mathbb{C}$ solves many of our woes with respect to factoring out roots. In particular, we have the following theorem, whose proof we omit but is not beyond your abilities to find:

Theorem 2.2. *(The Fundamental Theorem of Algebra:) Every complex polynomial $p(z)$ with degree n has n (possibly repeated) roots in the complex plane. In other words, we can factor every degree n polynomial into n complex roots: i.e. we can always find constants such that*

$$p(z) = C \cdot \prod_{k=1}^{n} (z - r_i).$$

As it turns out, there is a far stronger analogue to this theorem, which says (basically) that we can factor not just polynomials, but entire power series into their roots! This theorem is incredibly difficult to prove (you could easily spend a pair of quarters on complex analysis and not get to it,) so we state it without proof below:

Theorem 2.3. *Weierstrass Factorization Theorem: every complex power series $f(x) = \sum a_n z^n$ can be written in the form*

$$e^{g(z)} x^n \cdot \prod_{\text{all roots } r_k \text{ of } f} \left(1 - \frac{z}{r_k}\right),$$

for some natural number n and some other complex power series $g(x)$.

Basically, this says that we can separate any complex power series into its roots (the $(1 - \frac{z}{r_i})$ - parts and the z^k part), times some $e^{g(z)}$-part that's never 0.

One particular result you can derive from this theorem is the following factorization of $\sin(z)$:

$$\sin(z) = z \cdot \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{\pi n}\right).$$

The proof of this theorem – or indeed just that $\sin(z)$ can be written in the form above, if you tried to do this directly without the theorem! – are far beyond the scope of this course. But, even without the proofs, this should hopefully feel at least like a plausible result; after all, if we can factor out the roots for polynomials, then we ought to be able to do so for "infinte polynomials" like power series.

(A quick aside: for those of you who haven't seen it before, the **infinite product** of some sequence a_n, $\prod_{n=1}^{\infty} a_n$, is just defined by the limit

$$\lim_{N \to \infty} \prod_{n=1}^{N} a_n.$$

This should look like definition we used for an infinte series; it's the same idea, except with multiplication in place of addition.)

Using this, we can finally prove something we claimed back in week 2 of our course:

Theorem 2.4. $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof. So: recall our earlier-mentioned *deus ex machina* result that $\sin(z)$ could be "factored into its roots" – i.e that

$$\sin(z) = z \cdot \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{\pi n}\right).$$

We can rewrite this expression as the product

$$\sin(z) = z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{\pi n}\right) \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z}{\pi n}\right),$$

and bring terms together to further simplify this into the equation

$$\begin{aligned}\sin(z) &= z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{\pi n}\right) \cdot \left(1 + \frac{z}{\pi n}\right) \\ &= z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right);\end{aligned}$$

Thus, from the above, we know that we can write

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right).$$

Ok, so enough simplification. Why do we do this? Well: we also have the power series expansion $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots$, which tells us that

$$\begin{aligned}\frac{\sin(z)}{z} &= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots}{z} \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \ldots\end{aligned}$$

So these two quantities are the same! In particular, we know that they must share the same power series; consequently, these two objects must share the same z^2-coefficient. For the power series expression, finding this coefficient is easy – it's just $-\frac{1}{3!}$.

For the product, it's not much harder. Look at

$$\left(1 - \frac{z^2}{\pi^2 1^2}\right) \cdot \left(1 - \frac{z^2}{\pi^2 2^2}\right) \cdot \left(1 - \frac{z^2}{\pi^2 3^2}\right) \cdot \left(1 - \frac{z^2}{\pi^2 4^2}\right) \cdot \ldots.$$

How can we get a term involving z^2 out of such a product? Well; think way back, to the days of FOIL. How do we figure out what a complicated product of polynomials, like

$$(a_0 + a_1 z + \ldots a_q z^q) \cdot (b_0 + \ldots b_r z^r) \cdot (c_0 + \ldots + c_s z^t)$$

is? Well: all we do is just pick a term in the first polynomial – say, some $a_i z^i$ – and multiply it by some term in the second polynomial – say, $b_k z^k$ – and finally multiply it by some term in the third polynomial – say $c_l z^l$. If we do this exactly once for every single possible way of choosing terms out of these three polynomials, and add them up, this gives us the product of the polynomials! Essentially, this is just FOIL writ large.

In the infinte case it's just the same process! In order to figure out what the terms of

$$\left(1-\frac{z}{\pi^2 1^2}\right)\cdot\left(1-\frac{z}{\pi^2 2^2}\right)\cdot\left(1-\frac{z}{\pi^2 3^2}\right)\cdot\left(1-\frac{z}{\pi^2 4^2}\right)\cdot\ldots$$

are, we just need to look at the various terms we get by choosing one value from each $(1-\frac{z^2}{\pi^2 n^2})$ and multiplying them all together. In particular, if we're looking at the z^2 coefficient, the only terms that will have a z^2 as their coefficient are those that choose precisely one $\frac{z^2}{\pi^2 n^2}$ out of our giant product, and choose 1's the rest of the time! So, in short, we have that the z^2 terms are simply all of the fractions $-\frac{1}{\pi^2 n^2}$; so the z^2-coefficient is just

$$\sum_{n=1}^{\infty} -\frac{1}{\pi^2 n^2}.$$

Setting this equal to $-\frac{1}{3!}$ tells us that

$$-\frac{1}{3!} = \sum_{n=1}^{\infty} -\frac{1}{\pi^2 n^2}$$

$$\Rightarrow \qquad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

□

3 Calculus on the Complex Numbers

In Ma1a, pretty much the first thing we started to do with the real numbers, after building up their structure, was calculus: throughout the last quarter, we studied the concepts of limits, derivatives, and integrals, and this quarter we've been studying sequences and series. A natural thing to want to study here, then, is calculus on the **complex** numbers: i.e. what is the notion of a limit for complex numbers? What is a derivative of a complex-valued function?

We define these concepts here. Notice that in the following definitions, we're pretty much using the exact same definitions as in the real-valued case, just with complex values where we used to have real numbers:

Definition 3.1. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers converges to some value L if

$$\lim_{n\to\infty} |a_n - L| = 0.$$

To illustrate this definition's use, we work a quick example:

Example 3.2. The limit

$$\lim_{n\to\infty} e^{in}$$

fails to converge.

Proof. Consider the distance between two consecutive terms, $|e^{i(n+1)} - e^{in}|$. This is equal to

$$\begin{aligned} |e^{i(n+1)} - e^{in}| &= |e^{in}(e^i - 1)| \\ &= |e^{in}| \cdot |e^i - 1|. \end{aligned}$$

From before, we know that e^{in} is just the unit-length vector leaving the origin that makes an angle of n radians with the positive-real-axis. In specific, we can see that it is length 1, which leaves us with

$$|e^{i(n+1)} - e^{in}| = |e^i - 1|.$$

This is precisely the length of a chord corresponding to an angle of 1 radian in a radius 1 circle, which is (via formulas you can look up) $2\sin\left(\frac{1}{2}\right)$. In specific, this is a fixed nonzero constant: therefore, the terms of this sequence are not getting closer to each other as n goes to infinity. Therefore, we can conclude that they do not have a limit, as in order for a sequence to have a limit its terms must get arbitrarily close to each other as n goes to infinity. □

This definition of limits allows us to work with the idea of complex-valued series. Pretty much the only tool that we'll need to study complex-valued series is the idea that **absolute convergence ⇒ convergence**, whose statement is identical to the theorem we stated for real-valued series:

Definition 3.3. (Absolute convergence ⇒ convergence:) Take a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers. Then, if the real-valued series

$$\sum_{n=1}^{\infty} |a_n|$$

converges, so must the series

$$\sum_{n=1}^{\infty} a_n.$$

Using this, we can talk about complex power series!

Definition 3.4. A complex power series around the point c is simply a complex valued function $f(z)$ of the form $\sum_{n=0}^{\infty} a_n(z-c)^n$. Usually, we will study complex power series only in the case when $c = 0$, in which case our power series looks like $\sum_{n=0}^{\infty} a_n z^n$.

Our definitions for complex convergence, series, and power series look fairly similar to the ones we had for real series and power series; so, we might hope that some of our theorems for power series carry through. Thankfully, many of them do, with special attention to the following theorem:

Theorem 3.5. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a complex power series that converges for some $z_0 \in \mathbb{C}$, then for any $a \leq |z_0|$, we have that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on the circle of radius a in $\mathbb{C}$.*

This theorem has a lot of beautiful results! In particular, it tells us **why** we use the word radius in the phrase "radius of convergence:" this is because in the complex plane, the radius of convergence is an actual value r such that our power series converges for everything smaller in magnitude than r and diverges for everything in magnitude greater than r! In other words, power series on the complex plane either converge only at 0, on all of $\mathbb{C}$, or only inside some disk with some radius r (where the boundary points with magnitude r may or may not converge.)

To illustrate this picture, consider the following example:

Example 3.6. Find the radius of convergence of the Taylor series for $\frac{1}{1+x^2}$, as considered as a complex-valued power series

Proof. Recall from our earlier lectures that

$$T\left(\frac{1}{1+x^2}\right) = 1 - x^2 + x^4 - x^6 + x^8 \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Therefore, if we look at this series at some value $r \in \mathbb{R}^+$, we can use the idea that absolute convergence $\Rightarrow$ convergence along with the ratio test to see that because

$$\lim_{n\to\infty} \frac{|r|^{2n}}{r^{2(n-1)}} = r^2,$$

whenever $r < 1$ we have that this series converges. As well, at $r = 1$ our series is just

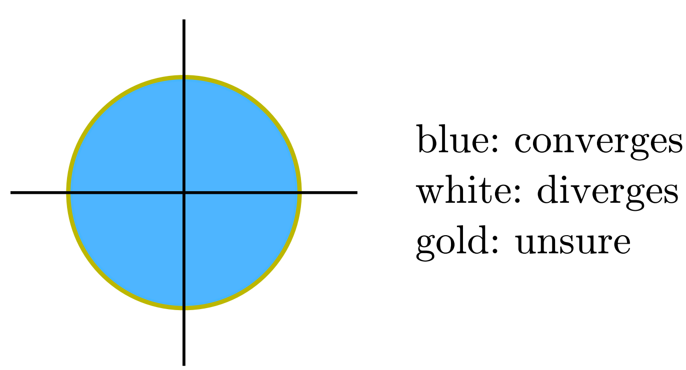
$$1 - 1 + 1 - 1 + 1 - 1 \ldots,$$

which clearly does not converge (as its partial sums oscillate between 1 and 0.) Therefore, by our theorem on radii of convergence, we know that the complex power series

$$\sum_{n=0}^{\infty} (-1)^n z^{2n}$$

must converge on all values of z with $|z| < 1$ (because for any $|z| < 1$, it converged on a value of $r > |z|$,) and diverge on all values of z with $z > 1$ (because it diverged at some value with norm 1.) In other words, its radius of convergence is 1.

To visually illustrate why this is a *radius* of convergence, we graph what we've just proven:



□

This, hopefully, should demonstrate how we attack pretty much all of these problems: to study the radius of convergence of this complex power series, all we had to do was look at its values on the real line. In general, because of our theorem on radii of convergence, this will always work! In other words, our mastery of real-valued power series will allow us to deal with complex-valued power series without much more effort.

The last topic we turn to in these lectures is the notion of **derivative**:

Definition 3.7. We define the derivative of a complex function $f : \mathbb{C} \to \mathbb{C}$ at a point z by

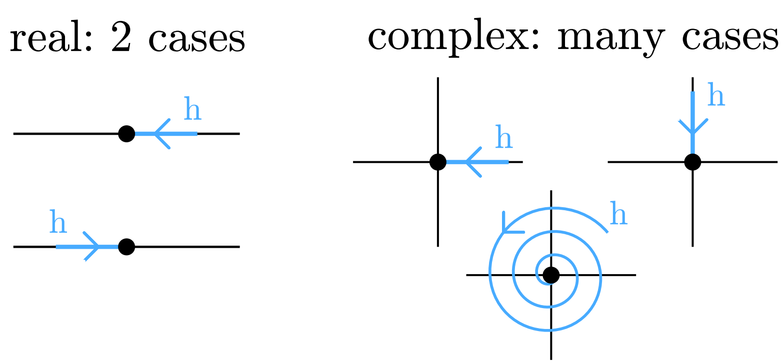
$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

where h is a complex number in the limit above.

This, as you may have noticed, looks identical to the definition we had for the derivative of a real function f(x):

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The only difference between the two is that in the real derivative, h is restricted to real values, where in the complex derivative h has far more choices! We illustrate this below:



In the real case, we're examining the limit $\lim_{h \to 0}$, where h is a real number; so, realistically, there are only two "paths" that we have to consider for studying this limit, $\lim_{h \to 0^-}$ and $\lim_{h \to 0^+}$. In the complex case, however, we have to deal with the limit $\lim_{z \to 0}$, where z is a complex number; in this case, we have infinitely many paths that might crop up.

As you might expect from this additional complication, this might make having a complex derivative a harder thing than having a real-valued derivative. The following, perhaps surprising, example shows how this comes up in practice:

Example 3.8. Consider the following complex-valued function:

$$f(x + iy) = x - iy,$$

i.e. $f(z) = \overline{z}$, the conjugate function. This function is not differentiable anywhere.

Proof. Take any value $z = x + iy$, and denote $h = a + ib$. Then, if we look at the definition of the derivative, we have

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \\ &= \lim_{a,b\to 0} \frac{f(x+a+iy+ib)-f(x+iy)}{a+ib} \\ &= \lim_{a,b\to 0} \frac{x+a-iy-ib-(x-iy)}{a+ib} \\ &= \lim_{a,b\to 0} \frac{a-ib}{a+ib}. \end{aligned}$$

If we look at this along the path $a = 0, b \to 0$, we have that this is the limit

$$= \lim_{b\to 0} \frac{0-ib}{0+ib} = -1;$$

conversely, if we look at this along the path $b = 0, a \to 0$ we have the limit

$$= \lim_{a\to 0} \frac{a-i\cdot 0}{a+i\cdot 0} = 1.$$

Because these values disagree, we know that there is no consistent limit as $h \to 0$ for this derivative to take on. $\square$

This is... really surprising. In the real numbers, it is **really hard** to construct a function that's continuous but not differentiable; here, we found one pretty much immediately. That said, a few of the basic theorems on differentiation still go through:

- $\frac{d}{dz}(f(z)) = 0$, if $f(z)$ is a constant.
- $\frac{d}{dz}(z) = 1$.
- $\frac{d}{dz}(f(z)+g(z)) = \left(\frac{d}{dz}f(z)\right) + \left(\frac{d}{dz}g(z)\right)$.
- $\frac{d}{dz}(f(z)\cdot g(z)) = \left(\frac{d}{dz}f(z)\right)g(z) + \left(\frac{d}{dz}g(z)\right)f(z)$: i.e. the product rule
- $\frac{d}{dz}(f(g(z))) = \left(\frac{d}{dz}f(z)\right)\Big|_{g(z)} \cdot \left(\frac{d}{dz}g(z)\right)$: i.e. the chain rule.
- If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a complex power series with radius of convergence r, then for any $z \in \mathbb{C}$ with $|z| < r$, we have

  $$\left(\frac{d}{dz}g(z)\right)(f(z)) = \sum_{n=1}^{\infty} a_n \cdot nz^{n-1}.$$

  In other words, power series still play well with derivatives whenever they're uniformly convergent, just like in the reals.

For example, we still have that $\frac{d}{dz}(z^n) = nz^{n-1}$, by just applying the product rule and the property that $\frac{d}{dz}(z) = 1$. As well, we still have that $\frac{d}{dz}(e^z) = e^z$ and $\frac{d}{dz}(\sin(z)) = \cos(z)$, $\frac{d}{dz}(\cos(z)) = -\sin(z)$ by using our observation that derivatives and power series "play nicely" with each other.

Integration is a somewhat stranger concept, as (unlike the case in $\mathbb{R}$) we suddenly have an entire **plane** to integrate functions over, instead of just a line! As this course is not a multivariable calculus course, we will omit discussing just what a complex integral might be; see me, however, if you're curious and want to read about them!