Math 1d — Instructor: Padraic Bartlett

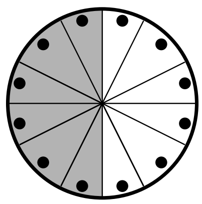
Lecture 3: Series with Negative Terms; Sequences of Functions

Week 4 — *Caltech - Winter 2012*

1 Random Questions

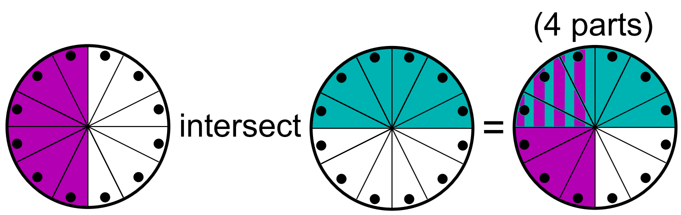
Question 1.1. *(This is a research question I've been looking at for a week or two. If you get somewhere, tell me!)*

Let A be a set with $4n$ elements in it. A ***halving*** *of A is a way to pick precisely half of the elements in A. A handy way to visualize a halving is illustrated below:*

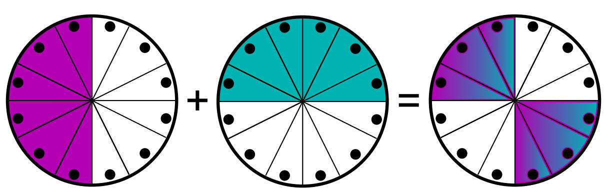


Here, we've arranged the points of A in a circle: with this visualization, we can think of a halving as a way to pick out "half" of the wedges around our circle, omitting the rest.

We say that two halvings are ***orthogonal*** *if overlapping both of them yields a quartering, i.e. a way to divide A into four equal parts, as depicted below:*



If this is true, we define the ***sum*** *of two orthogonal halvings H_1, H_2 as the halving H_{12} created as follows: let the chosen elements of H_{12} be precisely those that either* ***both*** *H_1 and H_2 have chosen, along with those that* ***neither*** *H_1 nor H_2 have chosen. An example is illustrated below:*



So: that's the setup. The question is now the following: suppose you take the collection of all halvings of a set A on $4n$ vertices, for $n \geq 2$, and assign to each halving a color R or B. Show that there are a pair of halvings H_1, H_2 such that H_1, H_2 and $H_{1,2}$ all have the same color.

Question 1.2. *Consider the function*

$$F(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n \cdot \pi \cdot x)}{10^n}$$

Show that this function is continuous everywhere. Show that it's differentiable nowhere.

2 Series with Negative Terms

Last week, we studied series with positive terms, and came up with a collection of tests (the ratio, integral, and comparison tests) that we could use to determine whether they converged or not. However, we often will want to study series with **negative** terms in them: how can we apply our old tests to such series? Also, do we have any new tools for dealing with series with both positive and negative terms, like $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$?

As it turns out, we do! Consider the following pair of results:

1. **Alternating series test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers such that

   - $\lim_{n\to\infty} a_n = 0$ monotonically, and
   - the a_n's alternate in sign, then

   the series $\sum_{n=1}^{\infty} a_n$ converges.

   When to use this test: when you have an alternating series.

2. **Absolute convergence** $\Rightarrow$ **convergence**: Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence such that

   $$\sum_{n=1}^{\infty} |a_n|$$

   converges. Then the sequence $\sum_{n=1}^{\infty} a_n$ also converges.

   When to use this test: whenever you have a sequence that has positive and negative terms, that is not alternating. (Pretty much every other test requires that your sequence is positive, so you'll often apply this test and then apply one of the other tests to the series $\sum_{n=1}^{\infty} |a_n|$.)

We illustrate the use of these two tests here:

Claim 1. (Alternating series test): The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges.

Proof. The terms in this series are alternating in sign: as well, they're bounded above and below by $\pm\frac{1}{n}$, both of which converge to 0. Therefore, we can apply the alternating series test to conclude that this series converges. □

Claim 2. (Absolute convergence $\Rightarrow$ convergence): The series

$$\sum_{n=1}^{\infty} \frac{\cos^n(nx)}{n!}$$

converges.

Proof. We start by looking at the series composed of the absolute values of these terms:

$$\sum_{n=1}^{\infty} \frac{|\cos^n(nx)|}{n!}$$

Because $|\cos(x)| \leq 1$ for all x, we can use the comparison test to notice that this series will converge if the series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

converges.

We can study this series with the ratio test:

$$\lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n-1)!}} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

which is less than 1. Therefore this series converges, and therefore (by the comparison test + absolute convergence $\Rightarrow$ convergence) our original series

$$\sum_{n=1}^{\infty} \frac{\cos^n(nx)}{n!}$$

converges. □

2.1 Rearranging Sums

We make a quick detour here to some more philosopical questions. Think, for a moment about what series **are**: just infinite sums of things! So: with finite sums, we know that addition has several nice properties. One particularly nice property that addition has is that it's **commutative**: i.e. the order in which we add things up doesn't matter! In other words, we know that

$$1 + 2 + 3 = 3 + 2 + 1 = 6.$$

A natural question we could ask, then, is the following: does this hold true with series? In other words, if we rearrange the terms in the series (say)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

will it still sum up to the same thing?

Well: let's try! Specifically, consider the following way to rearrange our series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots \\ &=^? \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) \dots\end{aligned}$$

In the second rearrangement, we've ordered terms in the following groups:

$$\ldots + \left(\frac{1}{\text{odd number}} - \frac{1}{2\cdot\text{that odd number}} - \frac{1}{2\cdot\text{that odd number}+2}\right) + \ldots$$

Notice that every term from our original series shows up in exactly one of these groups. Specifically, each odd number clearly shows up once: as well, for any even number, there are two cases: either it has exactly one factor of 2, in which case it's of the form (2·an odd number) and shows up exactly once, or it's a multiple of 4, in which case it shows up as a (2·an odd number+2), and also shows up once. So this is in fact a proper rearrangement! We haven't forgotten any terms, nor have we repeated any terms.

But, if we group terms as indicated below in our rearrangement, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{n} &=^? \left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)+\left(\frac{1}{7}-\frac{1}{14}-\frac{1}{16}\right)\ldots\\
&=\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}\ldots\\
&=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}\ldots\\
&=\frac{1}{2}\cdot\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\ldots\right)\\
&=\frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{n}.
\end{aligned}$$

So: if rearranging terms doesn't change the sum of an infinte series, we've just shown that

$$\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{n} = \frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{n}.$$

The only number that is equal to half of itself is 0: therefore, this series must sum to 0!

However: look at our series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ again. In specific, if we just expand this sum without rearranging anything, we can see that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}\ldots\\
&=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\left(\frac{1}{9}-\frac{1}{10}\right)+\ldots\\
&=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\frac{1}{7\cdot 8}+\frac{1}{9\cdot 10}+\ldots\\
&\geq\frac{1}{2}.
\end{aligned}$$

So this sum definitely **cannot** converge to 0, as all of its partial sums are $\geq \frac{1}{2}$! This answers our question earlier about rearranging series fairly definitively: we've just shown that rearranging series can do **unpredictable, terrible, and horrible** things to the series itself.

In fact, we have the following two theorems about what happens when series are rearranged:

Theorem 2.1. *Suppose that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. Then rearranging the a_n's does not change what our series converges to.*

Theorem 2.2. *Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series that is not absolutely convergent. Then for any $r \in \mathbb{R} \cup \{\pm\infty\}$, we can rearrange the a_n's so that the resulting series converges to r.*

Cool!

3 Sequences of Functions

Thus far, all of our discussions about convergence have dealt with real numbers: over the last three and a half weeks, we've developed a number of theorems and tests designed to let us know when various sequences and series of real numbers converge, and to tell us what they converge to. However, the basic concept of convergence is just one of "distance" – essentially, the claim that a sequence converges to a value is just a way of saying that its terms become very "close" to that value.

So: if the key idea of convergence is just this idea of "distance," can we extend this concept of convergence to other objects? In specific, can we extend the idea

First, note that by a **sequence** of functions we mean a collection $\{f_n\}_{n=1}^{\infty}$ of functions, indexed by the natural numbers. In this situation, suppose that all of the functions f_n are maps from some set A to the real numbers, and suppose further that we're given a function $f : A \to \mathbb{R}$. What could we possibly hope to mean by the equation

$$\lim_{n\to\infty} f_n = f \ ?$$

One possible idea would be to simply say that $\lim_{n\to\infty} f_n = f$ holds if and only if the sequences $\{f_n(x)\}$ converge to $f(x)$, for every $x \in A$. In other words, we have the following definition:

Definition 3.1. We say that a sequence $\{f_n\}$ of functions $A \to \mathbb{R}$ **converges pointwise** to some function $f : A \to \mathbb{R}$ if and only if $\lim_{n\to\infty} f_n(x) = f(x)$, for every $x \in A$.

So: if a sequence of real numbers all had a certain property – like all being positive, or greater than three, or integers – then if they converged to some value, that value often had to share that property. A natural question, then, is whether this holds true for sequences of functions; in other words, if we have a sequence of differentiable/continuous functions, must their pointwise limit be differentiable/continuous? If we have a sequence of functions all with integral 1 over some region, does their pointwise limit also have to have integral 1?

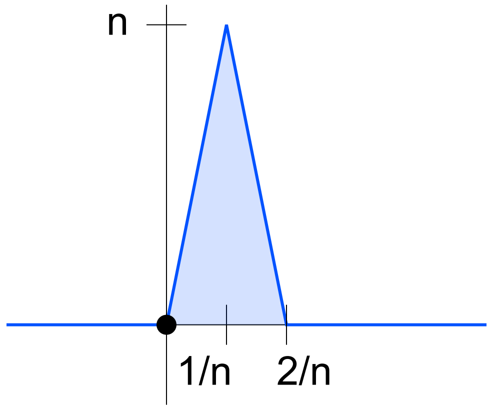
We answer these questions with the following two examples:

Example 3.2. Let

$$f_n(x) := \begin{cases} 1, & x \leq 0 \\ n^2 x, & 0 < x \leq 1/n \\ -n^2 x + 2n, & 1/n < x \leq 2/n \\ 0, & x \geq 2/n \end{cases}$$

What is the pointwise limit of the f_n's?

Proof. We start with a graph of the f_n's:



By construction, the integral of any of these f_n's is just the area of a triangle with base $2/n$ and height n – i.e. 1, for every f_n.

So: to calculate what these $f_n(x)$'s converge to, we break the x's apart into two cases:

1. $x \leq 0$. In this case, we have that
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0.$$
2. $x > 0$. In this case, we again know that we can find a value of N such that $\frac{2}{N} < x$; thus, for every $n > N$, we have that $f_n(x) = 0$, because $x > \frac{2}{n}$. Thus, we have that
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0.$$

Combining the results above tell us that the functions f_n converge pointwise to the function
$$f(x) := 0.$$

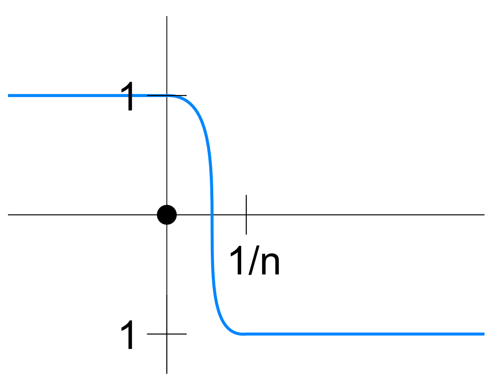
This means that, amongst other things, the integral is not preserved: while the integral of each f_n was 1, the integral of their limit f is just 0. □

So: what happens when we look at continuity/differentiability?

Example 3.3. Let
$$f_n(x) := \begin{cases} 1, & x \leq 0 \\ \cos(n\pi x), & 0 < x < 1/n \\ -1, & x \geq 1/n \end{cases}$$

What is the pointwise limit of the f_n's?

Proof. We start by graphing the f_n's:



Before beginning, we note that these functions indeed are all differentiable, as their derivatives on each part of their piecewise definition are

$$f_n'(x) := \begin{cases} 0, & x \leq 0 \\ -n\pi \cdot \sin(n\pi x), & 0 < x < 1/n \\ 0, & x \geq 1/n \end{cases},$$

and these all agree at the "cross-over" points 0 and $1/n$. So we're starting with continuous and differentiable functions: will we get a continuous/differentiable function in the limit?

To calculate what these $f_n(x)$'s converge to, we break the x's apart into two cases:

1. $x \leq 0$. In this case, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 1 = 1.$$

2. $x > 0$. In this case, we know (from the first quarter) that we can always find a value of N such that $\frac{1}{N} < x$; thus, for every $n > N$, we have that $f_n(x) = -1$, because $x > \frac{1}{n}$. Thus, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} -1 = -1.$$

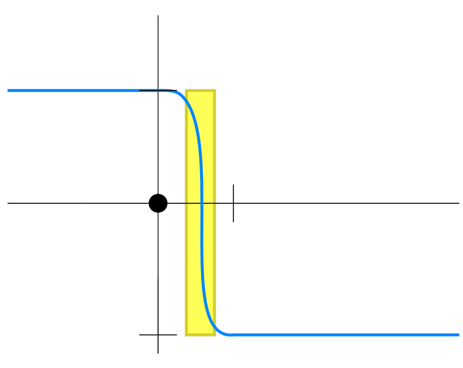
Combining the results above tell us that the functions f_n converge pointwise to the function

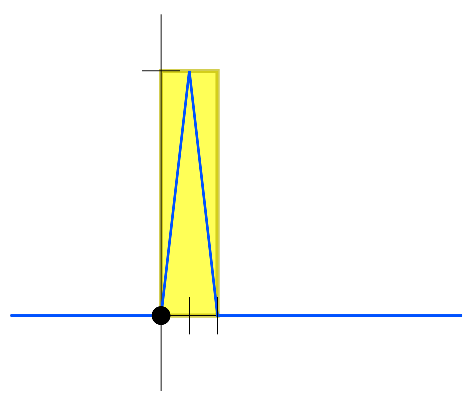
$$f(x) := \begin{cases} 1, & x \leq 0 \\ -1, & x > 0 \end{cases}$$

□

So: we started with a bunch of differentiable functions, and didn't even get something that's **continuous**.

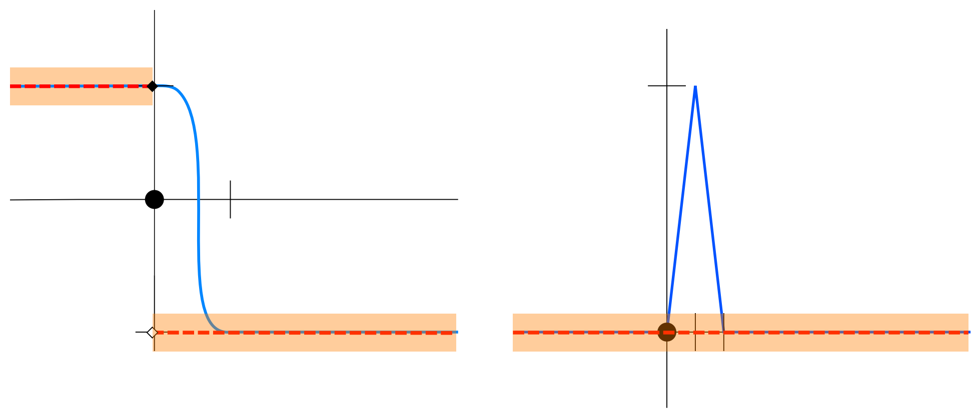
The moral of the above two examples seems to be that our notion of pointwise convergence, as intuitive and easy-to-use as it is, fails miserably at conserving most of the basic concepts we have for describing functions. Continuous functions fail to stay continuous, integrals aren't stable, differentiability has no hope; it's all a big mess. Yet, if we look at the graphs of the three "counterexamples" above, we might be able to come up with a fix for this problem:





In each of the two graphs above, there's a region (highlighted in yellow) where the graph seems to be almost moving "too fast" – i.e. while all of the f_n's remain continuous for every n, as the n's get large our functions begin to move very quickly in a very small area (as in the yellow regions.) So, while the f_n's converge pointwise to their pointwise limits f, throughout this convergence there is always a small region – corresponding to the yellow areas – where these functions were very **far** apart.

So: what if we used this as a new notion for convergence? I.e. what if we said that a sequence of functions f_n converge to a function f if and only if the f_n's become **uniformly** close to the function f? In other words: what if we said that $\lim_{n\to\infty} f_n = f$ if and only if the f_n's are eventually ϵ-close to f everywhere, for any epsilon and large enough n? Well, we definitely wouldn't have to worry about our two earlier examples, as the picture below shows:



Here, we can see that the f_n's are never completely contained within a small neighborhood (say, the one shaded in orange) of their pointwise limits f. So, while they do converge to f pointwise, they would fail to converge "uniformly" under our proposed definition above! Maybe there's some merit to this idea. Let's formally define this notion of a "uniform" convergence, and see where it takes us:

Definition 3.4. We say that a sequence $\{f_n\}$ of functions $A \to \mathbb{R}$ **converges uniformly** to some function $f : A \to \mathbb{R}$ if and only if for every $\epsilon > 0$, there is a N such that for every $n > N$,

$$|f(x) - f_n(x)| < \epsilon, \forall x \in A.$$

In other words, a sequence $\{f_n\}$ converges uniformly to some function f if and only if the f_n's are all $\epsilon-$close to f **everywhere**, for sufficiently large n.

It's worth noting the following proposition, which says that uniform convergence is a strictly stronger notion of convergence than pointwise convergence:

Proposition 3.5. *If a sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function f, then it must converge pointwise to f as well.*

Uniform convergence is, on its face, a "harder" definition to work with than pointwise convergence. The payoff for definition lies in the following three theorems, which state that uniform convergence preserves continuity, integrals, and (kinda) derivatives. We state them here:

Theorem 3.6. *If* $\lim_{n\to\infty} f_n = f$ *uniformly, and all of the functions* f_n, f *are integrable on some interval* $[a, b]$*, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Theorem 3.7. *If* $\lim_{n\to\infty} f_n = f$ *uniformly, and all of the functions* f_n *are continuous on some interval* (a, b)*, then so is* $f(x)$*.*

Theorem 3.8. *If the limit* $\lim_{n\to\infty} f_n = f$ *uniformly, and the limit* $\lim_{n\to\infty} f'_n$ *converges uniformly to some continuous function, then* f *is differentiable and* $\lim_{n\to\infty} f'_n(x) = f'(x)$*.*

To illustrate how to prove a sequence converges uniformly, consider the following example:

Example 3.9. Let

$$f_n(x) = \sum_{k=0}^{n} x^k.$$

Then the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ to the function $\frac{1}{1-x}$.

Proof. First, notice the identity

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x},$$

which you can prove by induction if you haven't seen it before. Using this identity, we can show that for any $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x},$$

because for any $|x| < 1$, $\lim_{n\to\infty} x^n = 0$.

So we've shown that the pointwise limit of the $f_n(x)$'s is $\frac{1}{1-x}$. We now want to show that this convergence is **uniform**: i.e. that for any distance $\epsilon > 0$, there is a cutoff point N past which the f_n's are all within this distance of their limit $\frac{1}{1-x}$.

Look at $\left|f_n - \frac{1}{1-x}\right|$:

$$\left|f_n - \frac{1}{1 - x}\right| = \left|\frac{1 - x^{n+1}}{1 - x} - \frac{1}{1 - x}\right| = \frac{|x|^{n+1}}{|1 - x|},$$

which for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ is greatest when $x = \pm\frac{1}{2}$: i.e. its maximum is

$$\frac{\frac{1}{2^{n+1}}}{\frac{1}{2}} = \frac{1}{2^n}.$$

So: we can make this quantity as small as we want! In other words, for any $\epsilon > 0$, we can pick a sufficiently large cutoff point N past which $\frac{1}{2^n} < \epsilon$: i.e. we can make the f_n's all arbitrarily close to $\frac{1}{1-x}$ everywhere.

Therefore, this sequence converges uniformly to $\frac{1}{1-x}$, as claimed. □

One immediate/awesome consequence of this is that taking integrals is now completely trivial!

Corollary 3.10. *If*

$$f(x) = \frac{1}{1-x}$$

then the antiderivative of $f(x)$ is the function

$$f(x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

Proof. We proved above that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

and that this convergence is uniform. Therefore, we know that integration commutes with limits: i.e. that

$$\int \frac{1}{1-x} dx = \int \left(\sum_{n=0}^{\infty} x^k \right) dx = \sum_{n=0}^{\infty} \left(\int x^k dx \right) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

More examples will come next week, when we start studying **power series!** $\square$