Math 1d	Instructor: Padraic Bartlett
Lecture 2: Series of Positive Numbers	
Week 2	*Caltech 2013*

1 Series of Positive Numbers

1.1 Series: Definitions and Tools

At the end of last Thursday's lecture, we introduced the Cauchy condition as our last "tool" for studying sequences, but didn't have time to discuss an example of its use before class ended. We start today's lecture with this example:

Definition 1.1. Cauchy sequences: We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m > n \geq N$, we have

$$|a_m - a_n| < \epsilon.$$

You can think of this condition as saying that Cauchy sequences "settle down" in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

Theorem 1. A sequence is Cauchy if and only if it converges.

The following sequence illustrates how we typically use this tool:

Claim 2. (Cauchy sequence example:) The sequence

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

converges.

Proof. To show that this sequence converges, we will use the Cauchy convergence tool, which tells us that sequences converge if and only if they are Cauchy.

How do we prove that a sequence is Cauchy? As it turns out, we can use a similar blueprint to the methods we used to show that a sequence converges:

- First, examine the quantity $|a_m - a_n|$, and try to come up with a very simple upper bound that depends on m and n and goes to zero. Example bounds we'd love to run into: $\frac{1}{n}, \frac{1}{mn}, \frac{1}{n}, \frac{1}{m^4 \log(n)}$. Things that won't work: $\frac{n}{m}$ (if n is really big compared to m, we're doomed!), $\frac{m}{n^{34}}$ (same!), 4.
- Using this upper bound, given $\epsilon > 0$, determine a value of N such that whenever $m > n > N$, our simple bound is less than ϵ.
- Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $m > n > N$, $|a_m - a_n| < \epsilon$.

Let's apply the above blueprint, and study $|a_m - a_n|$. Remember that we're assuming that $m > n$ here:

$$\begin{aligned} |a_m - a_n| &= \left| \sum_{k=1}^{m} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \right| \\ &= \sum_{k=n+1}^{m} \frac{1}{k^2} \end{aligned}$$

The following step may seem quite weird: it's motivated by partial fractions (because we want a way to simplify our $\frac{1}{k^2}$'s into simpler things), but it's mostly just an algebraic trick. The important thing is not to remember these tricks, but to just try tons of things until eventually *one* of them sticks:

$$\begin{aligned} \sum_{k=n+1}^{m} \frac{1}{k^2} &< \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\ &= \sum_{k=n+1}^{m} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ &= \sum_{k=n+1}^{m} \frac{1}{k-1} - \sum_{k=n+1}^{m} \frac{1}{k} \\ &= \sum_{k=n}^{m-1} \frac{1}{k} - \sum_{k=n+1}^{m} \frac{1}{k} \\ &= \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n}. \end{aligned}$$

This looks fairly simple!

Moving onto the second step: given $\epsilon > 0$, we want to force this quantity $\frac{1}{n} < \epsilon$. How can we do this? Well: if $m > n > N$, we have that $\frac{1}{n} < \frac{1}{N}$; so it suffices to pick N such that $\frac{1}{N} < \epsilon$.

Thus,we've shown that for any $\epsilon > 0$ we can find a N such that for any $m, n > N$,

$$|a_m - a_n| < \frac{1}{n} < \frac{1}{N} < \epsilon.$$

But this just means that our sequence is Cauchy! So, because all Cauchy sequences converge, we've proven that our sequence converges. $\square$

The example above is interesting for a number of reasons: not only was it a nice way to illustrate the Cauchy condition, it was the first example of a **series**! We define series here:

Definition 1.2. A sequence is called **summable** if the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums

$$s_n := a_1 + \ldots a_n = \sum_{k=1}^{n} a_k$$

converges.

If it does, we then call the limit of this sequence the **series** associated to $\{a_n\}_{n=1}^{\infty}$, and denote this quantity by writing

$$\sum_{n=1}^{\infty} a_n.$$

We say that a series$\sum_{n=1}^{\infty} a_n$ **converges** or **diverges** if the sequence $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ of partial sums converges or diverges, respectively.

Just like sequences, we have a collection of various tools we can use to study whether a given sequence converges or diverges:

1. **Comparison test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that $0 \leq a_n \leq b_n$, then the following statement is true:

   $$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

   When to use this test: when you're looking at something fairly complicated that either (1) you can bound above by something simple that converges, like $\sum 1/n^2$, or (2) that you can bound below by something simple that diverges, like $\sum 1/n$.

2. **Ratio test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

   $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

   then we have the following three possibilities:

   - If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
   - If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
   - If $r = 1$, then we have no idea; it could either converge or diverge.

   When to use this test: when you have something that is growing kind of like a geometric series: so when you have terms like 2^n or $n!$.

3. **Integral test**: If $f(x)$ is a function that is eventually[1] monotonically decreasing, then

   $$\sum_{n=N}^{\infty} f(n) \text{ converges if and only if } \int_{N}^{\infty} f(x)dx \text{ exists and is finite.}$$

To illustrate how to work with these definitions, we work a collection of examples here:

[1] A function is **eventually** monotonically decreasing if and only if there is some cutoff value N past which it is monotonically decreasing.

1.2 Series: Example Calculations

Claim 3. (Definition example 1:) The series

$$\sum_{n=1}^{\infty} n$$

diverges.

Proof. For any n, the sum of the numbers from 1 to n is just $\frac{n(n+1)}{2}$; therefore, by definition, we know that

$$\sum_{n=1}^{\infty} n = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k\right) = \lim_{n\to\infty}\frac{n(n+1)}{2} = \infty,$$

and therefore diverges. $\square$

Claim 4. (Definition example 2:) The **harmonic series**

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

diverges.

Proof. Notice the following inequalities:

$$\begin{aligned}
\left(\frac{1}{2}\right) &\geq \frac{1}{2} = \frac{1}{2}, \\
\left(\frac{1}{3}+\frac{1}{4}\right) &\geq 2\cdot\frac{1}{4} = \frac{1}{2}, \\
\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) &\geq 4\cdot\frac{1}{8} = \frac{1}{2}, \\
\left(\frac{1}{9}+\ \ldots\ +\frac{1}{16}\right) &\geq 8\cdot\frac{1}{16} = \frac{1}{2}, \\
&\vdots
\end{aligned}$$

In particular, notice that by applying these inequalities, we can show that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots \\
&\geq 1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots+\left(\frac{1}{2^{n-1}}+\ldots+\frac{1}{2^n}\right) \\
&\geq 1+\frac{1}{2}+\ldots+\frac{1}{2} \\
&= 1+\frac{n}{2},
\end{aligned}$$

for **any** value of n. Therefore, we know that this series must diverge, because there is no possible limit value L that is both finite and greater than $1+\frac{n}{2}$ for every value of n. $\square$

Claim 5. (Comparison test example): If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges, then the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$$

must also converge.

Proof. To see why, simply notice that each individual term $\frac{\sqrt{a_n}}{n}$ in this series is just the geometric mean of a_n and $\frac{1}{n^2}$. Because both of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are convergent, we would expect their average (for any reasonable sense of average) to also converge: in particular, we would expect to be able to bound its individual terms above by some combination of the original terms a_n and $\frac{1}{n^2}$!

In fact, we actually **can** do this, as illustrated below:

$$\begin{aligned} & 0 \leq \left(\sqrt{a_n} - \frac{1}{n}\right)^2 \\ \Rightarrow \qquad & 0 \leq a_n + \frac{1}{n^2} - 2\frac{\sqrt{a_n}}{n} \\ \Rightarrow \qquad & \frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right) < a_n + \frac{1}{n^2}. \end{aligned}$$

Look at the series $\sum_{n=1}^{\infty} \left(a_n + \frac{1}{n^2}\right)$. Because both of the series

$$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge, we can write

$$\sum_{n=1}^{\infty} \left(a_n + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and therefore notice that this series also converges; therefore, by the comparison test, our original series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ must also converge. $\square$

Claim 6. (Ratio test example): The series

$$\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^{n+1}}$$

converges.

Proof. Motivated by the presence of both a $n!$ and a 2^n, we try the ratio test:

$$\begin{aligned}
\frac{a_n}{a_{n-1}} &= \frac{\frac{2^n \cdot n!}{n^{n+1}}}{\frac{2^{n-1}\cdot(n-1)!}{(n-1)^n}} \\
&= \frac{2^n \cdot n! \cdot (n-1)^n}{2^{n-1}\cdot(n-1)!\cdot n^{n+1}} \\
&= \frac{2\cdot n\cdot(n-1)^n}{n^{n+1}} \\
&= \frac{2\cdot(n-1)^n}{n^n} \\
&= 2\cdot\left(\frac{n-1}{n}\right)^n \\
&= 2\cdot\left(1-\frac{1}{n}\right)^n
\end{aligned}$$

Here, we need one bit of knowledge that you may not have encountered before: the fact that the limit

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x,$$

and in particular that

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}.$$

(Historically, I'm pretty certain that that this is how e was defined; so feel free to take it as a definition of e itself.)

Applying this tells us that

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim n \to \infty 2\cdot\left(1-\frac{1}{n}\right)^n = \frac{2}{e},$$

which is less than 1. So the ratio test tells us that this series converges! □

Claim 7. (Integral test example): The series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

converges for $p > 1$, and diverges for $p \leq 1$.

Proof. Last week, we proved that the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges; as well, earlier today, we proved that the harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

diverges. So, by the comparison test, we know that for all $p \leq 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, and for all $p \geq 2$, this series converges. What about the values in between?

To study this, we use the integral test! First, notice that for any $p \in (1, 2)$, the function

$$f(x) = \frac{1}{x^p}, f : (0, \infty) \to (0, \infty)$$

is monotonically decreasing, because x^p is monotonically increasing and $f(x)$ is just the reciprocal of x^p. Therefore, by the integral test, we know that our series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ will converge precisely when the integral

$$\int_1^{\infty} \frac{1}{x^p} dx$$

exists and is finite.

We can calculate this integral easily, by just using the power rule $\frac{d}{dx} x^{1-p} = (1-p)x^{-p}$, which is valid for all $p \neq 1$, and integrating by using the antiderivative:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p} dx = \left.\frac{x^{1-p}}{1-p}\right|_1^{\infty} &= \left(\lim_{x \to \infty} \frac{x^{1-p}}{1-p} \right) - \frac{1^{1-p}}{1-p} \\
&= \left(\lim_{x \to \infty} \frac{1}{x^{p-1}(1-p)} \right) + \frac{1}{p-1} \\
&= 0 + \frac{1}{p-1}.
\end{aligned}$$

This integral exists and is finite for any $p > 1$; therefore, by the integral test and our earlier work with the comparison test, we've proven that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. $\square$