Math 1d — Instructor: Padraic Bartlett

Lecture 1: Sequences

Week 2 — *Caltech - Winter 2012*

1 Random Questions

A note on this section: at the start of lectures, I like to post interesting questions I've came across in my classes/research that are somewhat related to the material we're covering. These problems are strictly recreational! – I don't expect people to solve them, but offer them just to give you all something to think about. Let me know if you solve any of them, or want hints/solutions!

Question 1. Consider the following list of numbers:

$$\begin{aligned}&1,\\&11,\\&21,\\&1211,\\&111221,\\&312211,\ldots\end{aligned}$$

a.) There is a very simple rule that, given a entry on this list of numbers, will tell you what the next number is. What do you think it is? Using it, what is the next element of our list? (If you're stuck, look at problem 6 on the HW, where this sequence [the "look-and-say" sequence] is defined.)

b.) Show that the length of elements in the look-and-say sequence grows by about 1.3 characters per step: i.e. if l_n is the length of the n-th entry in the look-and-say sequence, show that

$$\lim_{n\to\infty} \frac{l_{n+1}}{l_n} \cong 1.3.$$

c.) Show that in specific, $\lim_{n\to\infty} \frac{l_{n+1}}{l_n} = \lambda$, Conway's constant, which is the unique root of the following degree-71 polynomial:

$$\begin{array}{ccccccccc}
x^{71} & -x^{69} & -2x^{68} & -x^{67} & +2x^{66} & +2x^{65} & +x^{64} & -x^{63} & -x^{62}- \\
x^{61} & -x^{60} & -x^{59} & +2x^{58} & +5x^{57} & +3x^{56} & -2x^{55} & -10x^{54} & -3x^{53}- \\
2x^{52} & +6x^{51} & +6x^{50} & +x^{49} & +9x^{48} & -3x^{47} & -7x^{46} & -8x^{45} & -8x^{44}+ \\
10x^{43} & +6x^{42} & +8x^{41} & -5x^{40} & -12x^{39} & +7x^{38} & -7x^{37} & +7x^{36} & +x^{35}- \\
3x^{34} & +10x^{33} & +x^{32} & -6x^{31} & -2x^{30} & -10x^{29} & -3x^{28} & +2x^{27} & +9x^{26}- \\
3x^{25} & +14x^{24} & -8x^{23} & -7x^{21} & +9x^{20} & +3x^{19} & -4x^{18} & -10x^{17} & -7x^{16}+ \\
12x^{15} & +7x^{14} & +2x^{13} & -12x^{12} & -4x^{11} & -2x^{10} & +5x^{9} & +x^{7} & -7x^{6}+ \\
7x^{5} & -4x^{4} & +12x^{3} & -6x^{2} & +3x & -6 & & &
\end{array}$$

Question 2. Consider the following sequence $\{a_n\}_{n=1}^\infty$, which we've written with some helpful line breaks:

$$\begin{aligned}
&0,\\
&-1, 0, 1\\
&-\frac{4}{2}, -\frac{3}{2}, -\frac{2}{2}, \ldots, \frac{2}{2}, \frac{3}{2}, \frac{4}{2},\\
&-\frac{9}{3}, -\frac{8}{3}, -\frac{7}{3}, \ldots, \frac{7}{3}, \frac{8}{3}, \frac{9}{3},\\
&-\frac{16}{4}, -\frac{15}{4}, -\frac{14}{4}, \ldots, \frac{14}{4}, \frac{15}{4}, \frac{16}{4}, \ldots\\
&\qquad\qquad\vdots
\end{aligned}$$

For which values of r in $\mathbb{R}$ can you find a subsequence of $\{a_n\}_{n=1}^\infty$ that converges to r?

2 Sequences and Series

This week's lectures in Math 1d are going to focus on **sequences** and **convergence**. A lot of the material here will feel like review; consequently, we're going to focus pretty heavily on examples and techniques. Before we can do that, however, we should review some of our basic definitions: what **is** a sequence? What does it actually mean for a sequence to converge? We review these definitions here:

2.1 Sequences: Definitions

Definition 2.1. A **sequence** of real numbers is a collection of real numbers $\{a_n\}_{n=1}^\infty$ indexed by the natural numbers.

Definition 2.2. A sequence $\{a_n\}_{n=1}^\infty$ is called **bounded** if there is some value $B \in \mathbb{R}$ such that $|a_n| < B$, for every $n \in \mathbb{N}$. Similarly, we say that a sequence is **bounded above** if there is some value U such that $a_n \leq U, \forall n$, and say that a sequence is **bounded below** if there is some value L such that $a_n \geq L, \forall n$.

Definition 2.3. A sequence $\{a_n\}_{n=1}^\infty$ is said to be **monotonically increasing** if $a_n \leq a_{n+1}$, for every $n \in \mathbb{N}$; conversely, a sequence is called **monotonically decreasing** if $a_n \geq a_{n+1}$, for every $n \in \mathbb{N}$.

Definition 2.4. Take a sequence $\{a_n\}_{n=1}^\infty$. A **subsequence** of $\{a_n\}_{n=1}^\infty$ is a sequence that we can create from the $\{a_n\}_{n=1}^\infty$'s by deleting some elements (making sure to still leave infinitely many elements left,) without changing the order of the remaining elements.

For example, if $\{a_n\}_{n=1}^\infty$ is the sequence

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots,$$

the sequences $0, 0, 0, 0, 0 \ldots$ and $1, 1, 1, 1, 1, \ldots$ are both subsequences of $\{a_n\}_{n=1}^\infty$, as is $0, 1, 0, 0, 0, 0, \ldots$ and many others.

Definition 2.5. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value λ if the a_n's "go to λ" at infinity. To put it more formally, $\lim_{n\to\infty} a_n = \lambda$ iff for any distance ϵ, there is some cutoff point N such that for any n greater than this cutoff point, a_n must be within ϵ of our limit λ.

In symbols:

$$\lim_{n\to\infty} a_n = \lambda \text{ iff } (\forall\epsilon)(\exists N)(\forall n > N)\ |a_n - \lambda| < \epsilon.$$

Convergence is one of the most useful properties of sequences! If you know that a sequence converges to some value λ, you know, in a sense, where the sequence is "going," and furthermore know where almost all of its values are going to be (specifically, close to λ.)

Because convergence is so useful, we've developed a number of tools for determining where a sequence is converging to:

2.2 Sequences: Convergence Tools

1. **The definition of convergence**: The simplest way to show that a sequence converges is sometimes just to use the definition of convergence. In other words, you want to show that for any distance ϵ, you can eventually force the a_n's to be within ϵ of our limit, for n sufficiently large.

   How can we do this? One method I'm fond of is the following approach:

   - First, examine the quantity $|a_n - L|$, and try to come up with a very simple upper bound that depends on n and goes to zero. Example bounds we'd love to run into: $1/n, 1/n^2, 1/\log(\log(n))$.
   - Using this simple upper bound, given $\epsilon > 0$, determine a value of N such that whenever $n > N$, our simple bound is less than ϵ. This is usually pretty easy: because these simple bounds go to 0 as n gets large, there's always some value of N such that for any $n > N$, these simple bounds are as small as we want.
   - Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $n > N$, $|a_n - L| < \epsilon$.

2. **Arithmetic and sequences**: These tools let you combine previously-studied results to get new ones. Specifically, we have the following results:

   - *Additivity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n + b_n = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$.
   - *Multiplicativity of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, then $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.
   - *Quotients of sequences*: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist, and $b_n \neq 0$ for all n, then $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$.

3. **Continuity and sequences**: This tool lets us use our knowledge of continuous functions to help evaluate series. Specifically, we have the following two claims

   - *Composing sequences and continuous functions*: if $\lim_{n\to\infty} a_n$ exists and $f(x)$ is a continuous function, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.

- *Switching from discrete limits to continuous limits*: Suppose that the limit $\lim_{x\to\infty} f(x)$ exists and is equal to L, for some function $f(x)$ and real number L. Then the limit of the sequence $\lim_{n\to\infty} f(n)$ exists and is also equal to L.

4. **Monotone and bounded sequences**: if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above and nondecreasing, then it converges; similarly, if it is bounded above and nonincreasing, it also converges. If a sequence is monotone, this is usually the easiest way to prove that your sequence converges, as both monotone and bounded are "easy" properties to work with. One interesting facet of this property is that it can tell you that a sequence converges without necessarily telling you what it converges to! So, it's often of particular use in situations where you just want to show something converges, but don't actually know where it converges *to*.

5. **Subsequences and convergence**: if a sequence $\{a_n\}_{n=1}^{\infty}$ converges to some value L, all of its subsequences must also converge to L.

   One particularly useful consequence of this theorem is the following: suppose a sequence $\{a_n\}_{n=1}^{\infty}$ has two distinct subsequences $\{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ that converge to different limits. Then the original sequence cannot converge! This is one of the few tools that you can use to directly show that something diverges, and as such is pretty useful.

6. **Squeeze theorem for sequences**: if $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n$ both exist and are equal to some value l, and the sequence $\{c_n\}_{n=1}^{\infty}$ is such that $a_n \leq c_n \leq b_n$, for all n, then the limit $\lim_{n\to\infty} c_n$ exists and is also equal to l. This is particularly useful for sequences with things like sin(horrible things) in them, as it allows you to "ignore" bounded bits that aren't changing where the sequence goes.

7. **Cauchy sequences**: We say that a sequence is **Cauchy** if and only if for every $\epsilon > 0$ there is a natural number N such that for every $m > n \geq N$, we have

$$|a_m - a_n| < \epsilon.$$

   You can think of this condition as saying that Cauchy sequences "settle down" in the limit – i.e. that if you look at points far along enough on a Cauchy sequence, they all get fairly close to each other.

   The Cauchy theorem, in this situation, is the following: a sequence is Cauchy if and only if it converges.

   The Cauchy criterion doesn't come up as often as the others in Math 1a (later in mathematics, however, it shows up pretty much everywhere!) Its main uses are for working with series (we'll have an example of this later, and define series later as well!), and for sequences whose limits we don't know: like the monotone-bounded-convergence theorem, this result doesn't need you to know where a sequence is converging to in order to show that it converges.

2.3 Sequences: Applications of Convergence Tools

In this section, we work an example for each of these tools. We start by illustrating how to prove a sequence converges using just the definition:

Claim 3. (Definition of convergence example:)

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0.$$

Proof. When we discussed the definition as a convergence tool, we talked about a "blueprint" for how to go about proving convergence from the definition: (1) start with $|a_n - L|$, (2) try to find a simple upper bound on this quantity depending on n, and (3) use this simple bound to find for any ϵ a value of N such that whenever $n > N$, we have

$$|a_n - L| < (\text{simple upper bound}) < \epsilon.$$

Let's try this! Specifically, examine the quantity $|\sqrt{n+1} - \sqrt{n} - 0|$:

$$\begin{aligned}
|\sqrt{n+1} - \sqrt{n} - 0| &= \sqrt{n+1} - \sqrt{n} \\
&= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&< \frac{1}{\sqrt{n}}.
\end{aligned}$$

All we did here was hit our $|a_n - L|$ quantity with a ton of random algebra, and kept trying things until we got something simple. The specifics aren't as important as the idea here: just start with the $|a_n - L|$ bit, and try everything until it's bounded by something simple and small!

In our specific case, we've acquired the upper bound $\frac{1}{\sqrt{n}}$, which looks rather simple: so let's see if we can use it to find a value of N.

Take any $\epsilon < 0$. If we want to make our simple bound $\frac{1}{\sqrt{n}} < \epsilon$, this is equivalent to making $\frac{1}{\epsilon} < \sqrt{n}$, i.e $\frac{1}{\epsilon^2} < n$. So, if we pick $N > \frac{1}{\epsilon^2}$, we know that whenever $n > N$, we have $n > \frac{1}{\epsilon^2}$, and therefore that our simple bound is $< \epsilon$. But this is exactly what we wanted!

In specific, for any $\epsilon > 0$, we've found a N such that for any $n > N$, we have

$$|\sqrt{n+1} - \sqrt{n} - 0| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \epsilon,$$

which is the definition of convergence. So we've proven that $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$. □

Claim 4. (Arithmetic and Sequences example:) The sequence

$$\begin{aligned}
a_1 &= 1, \\
a_{n+1} &= \sqrt{1 + a_n^2}
\end{aligned}$$

does not converge.

Proof. We proceed by contradiction: in other words, suppose that this sequence does converge to some value L, say. Then, examine the limit

$$\lim_{n\to\infty} a_n^2.$$

Because squaring things is a continuous operation, we know that

$$\lim_{n\to\infty} a_n^2 = (\lim_{n\to\infty} a_n)^2 = L^2.$$

However, we can also use the recursive definition of the a_n's to see that

$$\begin{aligned}\lim_{n\to\infty} a_n^2 &= \lim_{n\to\infty} \left(\sqrt{1+a_{n-1}^2}\right)^2 \\ &= \lim_{n\to\infty} (1+a_{n-1}^2)\end{aligned}$$

However, we know that $\lim_{n\to\infty} a_{n-1}^2 = \lim_{n\to\infty} a_n^2 = L^2$, because the two sequences are the same (just shifted over one place) and thus have the same behavior at infinity. Therefore, we know that both $\lim_{n\to\infty} 1$ and $\lim_{n\to\infty} a_{n-1}^2$ both exist: as a result, we can apply our result on arithmetic and sequences to see that

$$\lim_{n\to\infty} (1+a_{n-1}^2) = \left(\lim_{n\to\infty} 1\right) + \left(\lim_{n\to\infty} a_{n-1}^2\right) = 1 + L^2.$$

So, we've just shown that $L^2 = 1 + L^2$: i.e. $0 = 1$. This is clearly nonsense: so we've arrived at a contradiction. Therefore, our original assumption (that our sequence $\{a_n\}_{n=1}^{\infty}$ converged must be false – i.e. this sequence must diverge, as claimed. □

Claim 5. (Another arithmetic and sequences example:) For any two positive real numbers $x > y > 0$, show that

$$\lim_{n\to\infty} \frac{x^n - y^n}{x^n + y^n} = 1.$$

Proof. Using the fact that $0 < y < x$, write $y = cx$, for some positive real number $c < 1$. Then, our limit is just

$$\lim_{n\to\infty} \frac{x^n - (cx)^n}{x^n + (cx)^n} = \lim_{n\to\infty} \frac{x^n - c^n x^n}{x^n + c^n x^n} = \lim_{n\to\infty} \frac{x^n(1-c^n)}{x^n(1+c^n)} = \lim_{n\to\infty} \frac{1-c^n}{1+c^n}.$$

Now, notice that because $0 < c < 1$, $\lim_{n\to\infty} 1 - c^n = \lim_{n\to\infty} 1 + c^n = 1$. Because of this, we can move our limit above into the fraction (because both the top and bottom limits exist,) and get

$$\lim_{n\to\infty} \frac{1-c^n}{1+c^n} = \frac{\lim_{n\to\infty} 1 - c^n}{\lim_{n\to\infty} 1 + c^n} = \frac{1}{1} = 1.$$

So our original limit is 1, as claimed. □

Claim 6. (Continuity and Sequences example:) Evaluate the limit

$$\lim_{n\to\infty} n \cdot \left(e^{\frac{1}{n}} - 1\right)$$

Proof. Initially, it's not even clear what this limit converges to: while the n-part converges to infinity, the $(e^{1/n}-1)$ part goes to 0, and the behavior of their product is kinda confusing. So, how should we proceed?

Well: one thing we might be tempted to do is pass to the continuous case! In other words, consider instead the limit

$$\lim_{x\to\infty} x \cdot \left(e^{\frac{1}{x}} - 1\right).$$

If this limit exists, we know that it will be equal to our discrete limit $\lim_{n\to\infty} n \cdot \left(e^{\frac{1}{n}} - 1\right)$, by the tools we discussed earlier.

The advantage of doing this is now we can use tools like L'Hôpital's theorem to evaluate this limit, where before we had a discrete sequence (and couldn't even talk about things like taking derivatives!)

In specific, if we rewrite our limit as

$$\lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{1/x},$$

and substitute in $y = 1/x$, we have that our original limit is just

$$\lim_{y\to 0^+} \frac{e^y - 1}{y},$$

which we can just hit with L'Hôpital's rule (as both top and bottom go to 0) to get

$$\lim_{y\to 0^+} \frac{e^y}{1} = 1.$$

So our limit exists and is 1: therefore, our original discrete limit $\lim_{n\to\infty} n \cdot \left(e^{\frac{1}{n}} - 1\right)$ also exists and is 1. □

Claim 7. (Monotone convergence theorem example:) Let

$$a_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

Then the sequence $\{a_n\}_{n=1}^{\infty}$ converges.

Proof. As suggested above, let's try showing that this sequence is monotonically increasing and bounded to prove it converges.

Monotonically increasing is not hard to show: because the difference between a_{n+1} and a_n is

$$a_{n+1} - a_n = \sum_{k=0}^{n+1} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{(n+1)!},$$

which is positive, we know that $a_{n+1} > a_n$ for every n.

Bounded is not much harder. Take any term a_n, and expand it as the sum

$$a_n = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \ldots + \frac{1}{2 \cdot 3 \cdot \ldots \cdot n}.$$

How can we make this simpler, into something we can easily study and show is finite? Well: one way is to simply take the denominators of all of these fractions and replace all of the numbers greater than 2 with 2's. In other words, notice that

$$\begin{aligned} a_n &= 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \ldots + \frac{1}{2 \cdot 3 \cdot \ldots \cdot n} \\ &\leq 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2 \cdot 2} + \ldots + \frac{1}{2 \cdot 2 \cdot \ldots \cdot 2} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}}. \end{aligned}$$

But we know the sum on the right! In particular, by remembering our geometric sum identities from whenever they came up in high school (or proving them via induction, if you didn't see them before), we have

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}} = \frac{2^{n-1} - 1}{2^{n-1}} < 1.$$

So the entire sum is bounded above by $1 + 1 + 1 = 3$, for any n! So it's bounded above and monotonically increasing, and therefore convergent, via our theorem. □

Claim 8. (Subsequences example:) The sequence

$$\{a_n\}_{n=1}^{\infty} = 0, 1, 0, 1, 0, 1, 0, 1, \ldots$$

diverges.

Proof. Both

$$0, 0, 0, 0, 0, 0, \ldots$$

and

$$1, 1, 1, 1, 1, 1, \ldots$$

are subsequences of $\{a_n\}_{n=1}^{\infty}$. Therefore, because the first subsequence converges to 0 and the second subsequences converges to 1, which are distinct values, our tool tells us that the original sequence $\{a_n\}_{n=1}^{\infty}$ cannot converge, and thus must diverge. □

Claim 9. (Squeeze theorem example:)

$$\lim_{n \to \infty} \frac{\sin\left(n^2 \cdot \pi^{n^e - 12n} \cdot n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}\right)}{n} = 0.$$

Proof. The idea of squeeze theorem examples is that they allow you to get rid of awful-looking things whenever they aren't materially changing where the sequence is actually going. Specifically, in our example here, the sin(terrible things) part is awful to work with, but really isn't doing anything to our sequence: the relevant part is the denominator, which is going to infinity (and therefore forcing our sequence to go to 0.

Rigorously: we have that

$$-1 \leq \sin(\text{terrible things}) \leq 1,$$

no matter what terrible things we've put into the sin function. Dividing the left and right by n, we have that

$$-\frac{1}{n} \leq \frac{\sin(\text{terrible things})}{n} \leq \frac{1}{n},$$

for every n. Then, because $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, the squeeze theorem tells us that

$$\lim_{n\to\infty} \frac{\sin\left(n^2 \cdot \pi^{n^e - 12n} \cdot n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}\right)}{n} = 0$$

as well. □

Claim 10. (Cauchy sequence example:) The sequence

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

converges.

Proof. To show that this sequence converges, we will use the Cauchy convergence tool, which tells us that sequences converge if and only if they are Cauchy.

How do we prove that a sequence is Cauchy? As it turns out, we can use a similar blueprint to the methods we used to show that a sequence converges:

- First, examine the quantity $|a_m - a_n|$, and try to come up with a very simple upper bound that depends on m and n and goes to zero. Example bounds we'd love to run into: $\frac{1}{n}, \frac{1}{mn}, \frac{1}{n}, \frac{1}{m^4 \log(n)}$. Things that won't work: $\frac{n}{m}$ (if n is really big compared to m, we're doomed!), $\frac{m}{n^{34}}$ (same!), 4.

- Using this upper bound, given $\epsilon > 0$, determine a value of N such that whenever $m > n > N$, our simple bound is less than ϵ.

- Combine the two above results to show that for any ϵ, you can find a cutoff point N such that for any $m > n > N$, $|a_m - a_n| < \epsilon$.

Let's apply the above blueprint, and study $|a_m - a_n|$. Remember that we're assuming that $m > n$ here:

$$\begin{aligned} |a_m - a_n| &= \left| \sum_{k=1}^{m} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \right| \\ &= \sum_{k=n+1}^{m} \frac{1}{k^2} \end{aligned}$$

The following step may seem quite weird: it's motivated by partial fractions (because we want a way to simplify our $\frac{1}{k^2}$'s into simpler things), but it's mostly just an algebraic trick. The important thing is not to remember these tricks, but to just try tons of things until eventually *one* of them sticks:

$$\begin{aligned}
\sum_{k=n+1}^{m} \frac{1}{k^2} &< \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\
&= \sum_{k=n+1}^{m} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= \sum_{k=n+1}^{m} \frac{1}{k-1} - \sum_{k=n+1}^{m} \frac{1}{k} \\
&= \sum_{k=n}^{m-1} \frac{1}{k} - \sum_{k=n+1}^{m} \frac{1}{k} \\
&= \frac{1}{n} - \frac{1}{m} \\
&< \frac{1}{n}.
\end{aligned}$$

This looks fairly simple!

Moving onto the second step: given $\epsilon > 0$, we want to force this quantity $\frac{1}{n} < \epsilon$. How can we do this? Well: if $m > n > N$, we have that $\frac{1}{n} < \frac{1}{N}$; so it suffices to pick N such that $\frac{1}{N} < \epsilon$.

Thus,we've shown that for any $\epsilon > 0$ we can find a N such that for any $m, n > N$,

$$|a_m - a_n| < \frac{1}{n} < \frac{1}{N} < \epsilon.$$

But this just means that our sequence is Cauchy! So, because all Cauchy sequences converge, we've proven that our sequence converges. $\square$